Construction of Convergent Sequence in Cone 2-Normed Spaces

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***Abstract*—We introduce an idea of convergent sequence in a cone 2-normed space. We show that the convergence in 2-normed spaces using the definition of 2-norm by considering its dual space. Then we construct the convergence in cone 2-normed space, particularly for ℓ_2-space.**

***Index Terms*—Convergent sequence, cone 2-normed spaces.**

I. Introduction

Study of 2-normed space continues to grow. Among others, the study of 2-norm is done by linking its dual space [1], especially for the ℓ_2-space. With regards to the elaboration of the cone norm [2], [3], then the 2-norm was extended to cone 2-norm. Furthermore, the convergent properties in cone norm and 2-norm spaces have been studied in the literature. Then the construction of convergent sequence can be obtained on the cone 2-normed space, particularly for ℓ_2-space. Furthermore, to construct the convergence in cone 2-normed space, first we discuss the 2-norm, 2-normed cone spaces and convergence criterion in 2-norm as follows.

Definition 1 ([4]). *Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies:*

- (N1) $\|x, y\| \geq 0$ *for every* $x, y \in X$; $\|x, y\| = 0$ *iff* x *and* y *are linearly dependent;*
- (N2) $\|x, y\| = \|y, x\|$ *for every* $x, y \in X$;
- (N3) $\|x, \alpha y\| = |\alpha|\|x, y\|$ *for every* $x, y \in X$ *and* $\alpha \in \mathbb{R}$
- (N4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ *for every* $x, y, z \in X$.

The pair $(X, \|\cdot,\cdot\|)$ is called 2-normed space.

For history of inner product spaces and 2-normed spaces, we refer the interested reader to [4], [1], [5], [6], [7]. In [1], we have defined the 2-norm by linking its dual space with inner product $< x, z >$, which is formally defined as

$$\|x, y\| = \sup\left\{ \begin{vmatrix} < x, z > & < y, z > \\ < x, w > & < y, w > \end{vmatrix} ; z, w \in \ell^2 \; ; \|z\|, \|w\| \leq 1 \right\}$$

Furthermore for every $x_n, x, y \in \ell_2$, we obtain

$$\|x_n - x, y\| = \sup\left\{ \begin{vmatrix} < x_n - x, z > & < y, z > \\ < x_n - x, w > & < y, w > \end{vmatrix} ; z, w \in \ell^2 \; ; \|z\|, \|w\| \leq 1 \right\}$$

Therefore for $n \to \infty$, then

$$\sup\left\{ \begin{vmatrix} < x_n - x, z > & < y, z > \\ < x_n - x, w > & < y, w > \end{vmatrix} ; z, w \in \ell^2 \; ; \|z\|, \|w\| \leq 1 \right\} = 0$$

Moreover for $m, n \to \infty$, then

$$\sup\left\{ \begin{vmatrix} < x_m - x_n, z > & < y, z > \\ < x_m - x_n, w > & < y, w > \end{vmatrix} ; z, w \in \ell^2 \; ; \|z\|, \|w\| \leq 1 \right\} = 0$$

Definition 2. *A sequence (x_n) in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to $z \in X$ if for every $z \in X$ implies $\|x_n - x, z\| = 0$ for $n \to \infty$. Also, we say that (x_n) is Cauchy if for every $z \in X$ implies $\|x_m - x_n, z\| = 0$ for $m, n \to \infty$.*

Definition 3 ([3]). *A cone normed space is an ordered pair $(X, \|\cdot\|_C)$ where X is a linear space over $\mathbb{R}$ and $\|\cdot\|_C : X \to (E, P, \|\cdot\|)$ is a function satisfying:*

- (C1) $\|x\|_C \geq \theta$ *for every* $x \in X$;
- (C2) $\|x\|_C = \theta$ *if and only if* $x = 0$;
- (C3) $\|\alpha x\|_C = |\alpha|\|x\|_C$ *for every* $x \in X$ *and* $\alpha \in \mathbb{R}$;
- (C4) $\|x + y\|_C \leq \|x\|_C + \|y\|_C$ *for every* $x, y \in X$;

It is easy to see that $\mathbb{R}^n$ equipped with the standard Euclidean norm is a Banach space and if $P \subset \mathbb{R}^n$ for nonnegative $\mathbb{R}$, P is a cone.

Definition 4 ([3]). *Let X be a linear space over $\mathbb{R}$. Let $(E, \|\cdot\|)$ be a Banach space and $P \subset E$ be a cone. If the function $\|\cdot,\cdot\|_C : X \times X \to (E, P, \|\cdot\|)$ satisfies*

- (CN1) $\|x, y\|_C \succeq \theta$ *for every* $x, y \in X$; $\|x, y\|_C =$ *iff* x *and* y *are linearly dependent;*
- (CN2) $\|x, y\|_C = \|y, x\|_C$ *for every* $x, y \in X$;
- (CN3) $\|\alpha x, y\|_C = |\alpha|\|x, y\|_C$ *for every* $x, y \in X$ *and* $\alpha \in \mathbb{R}$;
- (CN4) $\|x, y + z\|_C \preceq \|x, y\|_C + \|x, z\|_C$ *for every* $x, y, z \in X$

then $(X, \|\cdot,\cdot\|_C)$ is called a cone 2-normed space.

II. Main Results

Let ℓ_2-space be a 2-normed space. A function $\|\cdot,\cdot\|_C : \ell_2 \times \ell_2 \to (\mathbb{R}^n, P, \|\cdot\|)$ defined by $\|x, y\|_C = \sum_{k=1}^{n} e_k \|x, y\|_{\ell_2}$, is a cone 2-normed space and we say an ℓ_2-space as a cone 2-normed space.

Theorem 1. *Let $(\ell_2, (\mathbb{R}^n, P, \|\cdot\|))$ be a cone 2-normed space with $\|x, y\|_C = \sum_{k=1}^{n} e_k \|x, y\|_{\ell_2}$ and let (x_m) be a sequence*

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in ℓ_2-space, then sequence (x_m) is convergent to $x \in \ell_2$ if for every $z \in \ell_2$, $\|x_m - x, z\|_C = \theta$ holds for $m \to \infty$.

Proof. The sequence (x_m) converges to $x \in \ell_2$ implies that for every $z \in \ell_2$, $\|x_m - x, z\|_{\ell_2} = 0$ holds for $m \to \infty$. Moreover for every $z \in \ell_2$, then $\|x_m - x, z\|_C = \sum_{k=1}^n e_k \|x_m - x, z\|_{\ell_2}$. Furthermore for $m \to \infty$ we have $\|x_m - x, z\| = \sum_{k=1}^n e_k \|x_m - x, z\|_{\ell_2} = \sum_{k=1}^n e_k.0 = \theta$. Therefore $\|z, x_m - x\|_C = \theta$ for $m \to \infty$ and for every $z \in \ell_2$. □

Theorem 2. *Let $(\ell_2, (\mathbb{R}^n, P, \|\cdot\|))$ be a cone 2-normed space with $\|x, y\|_C = \sum_{k=1}^n e_k \|x, y\|_{\ell_2}$ and let (x_m) be a sequence in ℓ_2-space, then sequence (x_m) is Cauchy if for every $z \in \ell_2$, $\|x_m - x, z\|_C = \theta$ holds for $m, n \to \infty$.*

Proof. The sequence (x_m) is Cauchy implies that for every $z \in \ell_2$, $\|x_m - x_n, z\|_{l_2} = 0$ holds for $m, n \to \infty$. Moreover for every $z \in \ell_2$, the following holds $\|x_m - x_n, z\|_C = \sum_{k=1}^n e_k \|x_m - x_n, z\|_{l_2}$. Furthermore for $m, n \to \infty$, we have $\|x_m - x_n, z\|_C = \sum_{k=1}^n e_k \|x_m - x_n, z\|_{\ell_2} = \sum_{k=1}^n e_k.0 = \theta$. Therefore $\|x_m - x_n, z\|_C = \theta$ for $m, n \to \infty$ and for every $z \in \ell_2$. □

Theorem 3. *Let $(\ell_2, (\mathbb{R}^n, P, \|\cdot\|))$ be a cone 2-normed space with $\|x, y\|_C = \sum_{k=1}^n e_k N_k \|x, y\|_{\ell_2}$ for a nonnegative real number N_k and let (x_m) be a sequence in ℓ_2-space, then sequence (x_m) is convergent to $x \in \ell_2$ if for every $z \in \ell_2$, $\|x_m - x, z\|_C = \theta$ for $m \to \infty$.*

Proof. The sequence (x_m) converges to $x \in \ell_2$ implies that for every $z \in \ell_2$, $\|x_m - x, z\|_{\ell_2} = 0$ for $m \to \infty$. Moreover for every $z \in \ell_2$, the following holds $\|x_m - x, z\|_C = \sum_{k=1}^n e_k N_k \|x_m - x, z\|_{\ell_2}$. Futhermore for $m \to \infty$, we have $\|x_m - x, z\|_C = \sum_{k=1}^n e_k N_k \|x_m - x, z\|_{\ell_2} = \sum_{k=1}^n e_k N_k.0 = \theta$. Therefore $\|z, x_m - x\|_C = \theta$ for $m \to \infty$ and for every $z \in \ell_2$. □

Theorem 4. *Let $(\ell_2, (\mathbb{R}^n, P, \|\cdot\|))$ be a cone 2-normed space with $\|x, y\|_C = \sum_{k=1}^n e_k N_k \|x, y\|_{\ell_2}$ for nonnegative real number N_k and let (x_m) be a sequence in ℓ_2-space, then sequence (x_m) is Cauchy if for every $z \in \ell_2$, $\|x_m - x_n, z\|_C = \theta$ for $m, n \to \infty$.*

Proof. The sequence (x_m) is Cauchy implies that for every $z \in \ell_2$, $\|x_m - x_n, z\|_{\ell_2} = 0$ for $m, n \to \infty$. Moreover for every $z \in \ell_2$, the following holds $\|x_m - x_n, z\|_C = \sum_{k=1}^n e_k N_k \|x_m - x_n, z\|_{\ell_2}$. Futhermore for $m, n \to \infty$, we have $\|x_m - x_n, z\|_C = \sum_{k=1}^n e_k N_k \|x_m - x_n, z\|_{\ell_2} = \sum_{k=1}^n e_k N_k.0 = \theta$. Therefore $\|x_m - x_n, z\|_C = \theta$ for $m, n \to \infty$ and for every $z \in \ell_2$. □

Theorem 5. *Let $(\ell_2, (\mathbb{R}^n, P, \|\cdot\|))$ be a cone 2-normed space and (x_m) be a sequence in ℓ_2 space, then sequence (x_m) is convergent to $x \in \ell_2$ if for every $z \in \ell_2$, $\|x_m - x, z\|_C = \theta$ for $m \to \infty$.*

Proof. The sequence (x_m) converges to $x \in \ell_2$ implies that for every $z \in \ell_2$, $\|x_m - x, z\|_{\ell_2} = 0$ holds for $m \to \infty$. Moreover for every $z \in \ell_2$, there exists a nonnegative real number C_k such that $\|x_m - x, z\|_C = \sum_{k=1}^n e_k C_k \|x_m - x, z\|_{\ell_2}$. Furthermore for $m \to \infty$, we have $\|x_m - x, z\|_C = \sum_{k=1}^n e_k C_k \|x_m - x, z\|_{\ell_2} = \sum_{k=1}^n e_k C_k.0 = \theta$. Therefore $\|z, x_m - x\|_C = \theta$ for $m \to \infty$ and for every $z \in \ell_2$. □

Theorem 6. *Let $(\ell_2, (\mathbb{R}^n, P, \|\cdot\|))$ be a cone 2-normed space and (x_m) be a sequence in l_2-space, then sequence (x_m) is Cauchy if for every $z \in \ell_2$, $\|x_m - x_n, z\|_C = \theta$ for $m, n \to \infty$.*

Proof. The sequence (x_m) is Cauchy implies for every $z \in \ell_2$, $\|x_m - x_n, z\|_{\ell_2} = 0$ for $m, n \to \infty$. Moreover for every $z \in \ell_2$, with nonnegative real number C_k such that $\|x_m - x_n, z\|_C = \sum_{k=1}^n e_k C_k \|x_m - x_n, z\|_{\ell_2}$. Furthermore for $m, n \to \infty$, we have $\|x_m - x_n, z\|_C = \sum_{k=1}^n e_k C_k \|x_m - x_n, z\|_{\ell_2} = \sum_{k=1}^n e_k C_k.0 = \theta$. Therefore $\|x_m - x_n, z\|_C \to \theta$ for $m, n \to \infty$ and for every $z \in \ell_2$. □

III. CONCLUSIONS

This work has developed convergence of a sequence in cone 2-normed spaces. This results can be extended to the convergence in n-normed space.

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