The Fuzzy Lattice of Ideals and Filters of an Almost Distributive Fuzzy Lattice

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Abstract—In this paper, the concept of fuzzy lattice is discussed. It is proved that a fuzzy poset \((I_1(L), B)\) and \((F_1(L), B)\) forms a fuzzy lattice, where \(I_1(L)\) and \(F_1(L)\) are the set containing all ideals, and the set containing all filters of an Almost Distributive Fuzzy Lattice(ADFL) respectively. In addition we proved that, a fuzzy poset \((PL_1(L), B)\) and \((PF_1(L), B)\) forms fuzzy distributive lattice, where \(PL_1\) and \(PF_1\) denotes the set containing all principal ideals and the set containing all principal filters of an ADFL. Finally, it is proved that for any ideal I and filter F of an ADFL, \(I_1 = \{i_L : i \in I\}\) and \(F_1 = \{f_L : f \in F\}\) are ideals of a fuzzy distributive lattice \((PL_1(L), B)\) and \((PF_1(L), B)\) respectively, and \(I_1 = \{f_L : f \in F\}\) and \(I_1 = \{i_L : i \in I\}\) are filters of a distributive fuzzy lattice \((PL_1(L), B)\) and \((PF_1(L), B)\) respectively.

Index Terms—Almost distributive fuzzy lattice, filters, fuzzy lattice, ideals.

I. INTRODUCTION

The axiomatization of Boole’s two valued propositional calculus led to the concept of Boolean Algebra and the class of Boolean Algebras(Ring). This includes the ring theoretic generalizations and the lattice theoretic generalizations like Heyting Algebras and distributive lattice. U.M.Samy and G.C.Rao [1] introduced the concepts of an ADL as a common abstraction of distributive lattice.

On the other hand, Zadeh [2] was the first mathematician to introduced the concepts of fuzzy and, to define and study fuzzy relations, Sanchez [3], Goguen [4] adapted this concept. The notion of partial order and lattice order goes back to 19th century investigations in logic. The concepts of fuzzy sublattices and fuzzy ideals of a lattice was introduced by Yuan and Wu [5]. Fuzzy lattice was defined as a fuzzy algebra by Ajmal and Thomas [6] and they characterized fuzzy sublattices for a first time. In 2009, fuzzy partial order relation was characterized minors of its level set by Chon [7]. Chon in the same paper defined a fuzzy lattice as a fuzzy relation, developed basic properties and characterized a fuzzy lattice by its level set.

As a continuation of these studies, in 2016 Berhanu [8] define an Almost Distributive Fuzzy Lattice as a generalization of Distributive Fuzzy Lattice and fuzzyfy some properties of the classical Almost Distributive Lattice using the fuzzy partial order relation and fuzzy lattice defined by Chon [7]. In the same year Berhanu and Bekalu [9], introduced the concepts of ideals and filters of an Almost Distributive fuzzy lattice. In addition, Berhanu and Bekalu [10] in the same year, introduced the concepts of principal ideals and filters of an ADFL as a continuation of [9].

Again, as a continuation of Berhanu and Bekalu’s [9] and [10] work, in this paper we introduce the concepts of fuzzy lattices by defining a fuzzy relation on the set containing all ideals of a given Almost Distributive Fuzzy Lattice, and on the set containing all filters of a given Almost Distributive Fuzzy Lattice. In addition, we proved that a fuzzy poset \((PL_1(L), B)\) and \((PF_1(L), B)\) forms a distributive fuzzy lattice, where \(PL_1\) is the set containing all principal ideals, and \(PF_1\) denotes the set containing all filters of a given ADFL L. Moreover in this paper, we introduce ideals and filters of the two distributive fuzzy lattice \((PL_1(L), B)\) and \((PF_1(L), B)\) induced by arbitrary ideal I and filter F of an ADFL L.

In this work, we will use Fuzzy Lattice defined by Chon [7], and an Almost Distributive Fuzzy Lattice introduced by Berhanu [8].

II. PRELIMINARIES

In this section we recall some definitions, basic and important results from [9],[10],[8], and [7] that will be required in this paper.

In this paper, an Almost Distributive Fuzzy Lattice \((R, A)\) is denoted by L, and R represents a non empty set such that L = \((R, A)\) forms an Almost Distributive fuzzy lattice(ADFL).

Definition 1 ([7]): Let X be a set. A function \(A : X \times X \rightarrow [0, 1]\) is called a fuzzy relation in X. The fuzzy relation A in X is reflexive if and only if \(A(x, x) = 1\) for all \(x \in X\). A is transitive if and only if \(A(x, z) \geq \text{sup}_{y \in X} \min(A(x, y), A(y, z))\), and A is antisymmetric if and only if \(A(x, y) > 0\) and \(A(y, x) > 0\) implies \(x = y\). A fuzzy relation A is fuzzy partial order relation if A is reflexive, antisymmetric and transitive. A fuzzy partial order relation A is a fuzzy total order relation if and only if \(A(x, y) > 0\) or \(A(y, x) > 0\) for all \(x, y \in R\). If A is a fuzzy partial order relation in a set X, then \((X, A)\) is called a fuzzy partially ordered set or a fuzzy poset.

Definition 2 ([8]): Let \((R, \lor, \land, 0)\) be an algebra of type \((2, 2, 0)\) and \((R, A)\) be a fuzzy poset. Then we call \(L = (R, A)\) is an Almost Distributive Fuzzy Lattice (ADFL) if the following axioms are satisfied:

1) \(A(a, a \lor 0) = A(a \lor 0, a) = 1;\)
2) \(A(0, 0 \land a) = A(0 \land a, 0) = 1;\)
3) \(A((a \lor b) \land c, (a \land c) \lor (b \land c)) = A((a \land c) \lor (b \land c), (a \lor b) \land c) = 1;\)
4) $A(a \land (b \lor c)) = A((a \land b) \lor (a \land c) \lor (b \land c)) = 1$ ;
5) $A((a \lor b) \land (a \lor c)) = A((a \lor b) \land (a \lor c) \land (b \lor c)) = 1$ ;
6) $A((a \lor b) \land (a \lor c) \land (b \lor c)) = 1$ ;

Definition 3 ([13]): Let $L$ be an ADFL. Then for any $a, b, c \in R$, $a \leq b$ if and only if $A(a, b) > 0$.

As a result of this definition, for any $a, b \in R$, $A(a, b) > 0$ if and only if $a \land b = a$ or $a \lor b = b$, and hence we have the following Theorem which are the main results of [8].

Theorem 4 ([8]): Let $L = (R, A)$ be an ADFL. Then for each $a, b$ and $c$ in $R$:
1) If $A(a, b) > 0$ then $A(b, a) = 0$, where $a \neq b$;
2) $A(a \land b, a) > 0$ and $A(b \land a, a) > 0$;
3) $A(a, b \land c) > 0$ and $A(b, a \land c) > 0$;
4) $A(a \land b, a) > 0$ and $A(b \land a, a) > 0$ whenever $A(a, b) > 0$;
5) $A(a \land b, b) > 0$ and $A(b \land a, a) > 0$ whenever $A(a, b) > 0$;
6) $A(0, a \land 0) > 0$ and $A(0, a \lor 0) > 0$;
7) $A(a, a \land 0) > 0$ and $A(a, a \lor 0) > 0$;
8) $A(1, a) \lor (a, a) = 1$ and $A(a, a) \lor (a, a) = 1$;
9) $A(a, a \lor (a, a)) = 1$ and $A(a, (a, a) \lor (a, a)) = 1$;
10) $A(a \land (a, a), a \lor (a, a)) = 1$;
11) $A((a \lor (a, a)), (a \lor (a, a))) = 1$;
12) $A(a \lor (a, a), a \lor (a, a)) = 1$;
13) $A(a \lor (a, a), a \lor (a, a)) = 1$;
14) $A((a \lor (a, a)), (a \lor (a, a))) = 1$;
15) $A((a \lor (a, a)), (a \lor (a, a))) = 1$;
16) $A((a \lor (a, a)), (a \lor (a, a))) = 1$.

Definition 5 ([19]): Let $L$ be an ADFL and let $I$ be any non-empty subset of $R$. $I$ is said to be an ideal of an ADFL $L$, if it satisfies the following axioms:
(i) $a, b \in I$ implies that $a \lor b \in I$
(ii) $a \in I$ and $b \in R$ implies that $a \land b \in I$.

Definition 6 ([19]): Let $L$ be an ADFL and let $F$ be any non-empty subset of $R$. Then $F$ is said to be filters of an ADFL $L$, if it satisfies the following axioms:
(i) $a, b \in F$ implies that $a \land b \in F$
(ii) $a \in F$ and $b \in R$ implies that $a \lor b \in F$.

Theorem 7 ([9]): Let $L$ be an ADFL and $S$ be any non-empty subset of $R$. Then, $\langle S \rangle_L = \{x \in R \mid (x, \cup_{i=1}^n s_i) > 0, \forall s_i \in S \}$ is the smallest ideal of $L$ containing $S$.

Theorem 8 ([9]): Let $L$ be an ADFL and $F$ be any non-empty subset of $R$. Then $\langle F \rangle_L = \{x \in R \mid (A_{\bigwedge 1}^{n} f_i, x \land (A_{\bigvee 1}^{n} f_i)) > 0, \forall f_i \in F \}$ is the smallest ideal of $L$ containing $F$.

Theorem 9 ([9]): Let $I$ and $J$ are two ideals of an ADFL $L$. Then $I \cap J$ and $I \cup J$ are also ideals of $L$, where $I \cap J = I \cap J$ and $I \cup J = \{x \in R \mid (x \land (x \in I \cap J) \land (x \in J) \}$.

Theorem 10 ([9]): Let $F$ and $G$ are two filters of an ADFL $L = (R, A)$. Then $F \land G$ and $G \land F$ are also filters of $L$, where $F \land G = F \land G$ and $F \land G = \{x \land (x \in F) \land (y \in G) \}$.

Definition 11 ([10]): Let $L$ be an ADFL and let $a \in R$. An ideal generated by a singleton set $\{a\}$ denoted by $[a]_A$ is a principal ideal of $L$. Similarly, a filter generated by a singleton set $\{a\}$ denoted by $[a]_A$ is a principal filter of $L$ generated by $a$.

Lemma 12 ([10]): Let $L$ be an ADFL and let $a \in R$. The set $\{x \in R \mid A(x, a) > 0\}$ and $\{x \in R \mid A(a, x) > 0\}$ are principal ideal and principal filter of $L$ generated by $a$, respectively.

Corollary 13 ([10]): Let $L$ be an ADFL and let $a \in R$. Then for any element $a$ of $R$, $a \in [a]_A$ and $a \in [a]_A$.

Corollary 14 ([10]): Let $L$ be an ADFL. For any $a, b \in R$,
(i) $[a]_A \setminus [b]_A = (a \lor b) \land [b]_A$;
(ii) $[a]_A \setminus [b]_A = [b]_A \land [b]_A$;
(iii) $[a]_A \setminus [b]_A = [b]_A \land [b]_A$;
(iv) $[a]_A \setminus [b]_A = [b]_A \land [b]_A$.

Theorem 15 ([10]): Let $L$ be an ADFL. Then for any $a, b \in R$, the followings are equivalent:
(i) $[a]_A \subset [b]_A$
(ii) $A(x, a \land b) > 0$, for all $x \in [a]_A$
(iii) $A(b \lor x, b) > 0$, for all $x \in [a]_A$
(iv) $[b]_A \subset [a]_A$.

III. THE FUZZY LATTICE OF IDEALS AND FILTERS OF AN ALMOST DISTRIBUTIVE FUZZY LATTICE

In this section we introduce the concept of a fuzzy lattice by defining a fuzzy relation B on the set containing all ideals $I_x(L)$, and on the set containing all filters $F_x(L)$ of a given Almost Distributive Fuzzy Lattice $L$. In addition, we proved that a fuzzy poset $(P_{IA}(L), B)$ and $(P_{IA}(L), B)$ forms a fuzzy distributive lattice, where $P_{IA}(L)$ is the set containing all principal ideals, and $P_{IA}(L)$ denotes the set containing all filters of a given ADFL $L$ respectively. Moreover in this section, we introduce ideals and filters of the two fuzzy distributive lattice $(P_{IA}(L), B)$ and $(P_{IA}(L), B)$ induced by an arbitrary ideal $I$ and filter $F$ of an ADFL $L$. In this paper, $I_x(L)$ and $F_x(L)$ represents the set containing all ideals and the set containing all filters of an ADFL $L$ respectively.

Definition 16 ([7]): Let $(X, A)$ be a fuzzy poset and let $B \subset X$. An element $u \in X$ is said to be an upper bound for a subset $Y$ if and only if $A(b, u) > 0$ for all $b \in Y$. An upper bound $u_0$ is the least upper bound of $Y$ if and only if $A(u_0, u) > 0$ for every upper bound $u$ of $Y$. An element $v \in X$ is said to be a lower bound for a subset $Y$ if and only if $A(v, b) > 0$ for all $b \in B$. A lower bound $v_0$ is the greatest lower bound of $Y$ if and only if $A(v, v_0) > 0$ for every lower bound $v$.

Consider a fuzzy relation $B$ on $I_x(L)$, and we have the following proposition.

Proposition 17: Let $L$ be an ADFL and $(I_x(L), R)$ be a fuzzy Poset. For any $I, J \in I_x(L)$, $B(I, J) > 0$ if and only if for all $i \in I$ there exists $j \in J$ such that $A(i, j) > 0$.

Proof: Let $I, J \in I_x(L)$. Suppose $B(I, J) > 0$, and let $i \in I$.

Since $I \subseteq J$, then $A(i, i) > 0$.

Conversely, suppose for all $i \in I$ there exists $j \in J$ such that $A(i, j) > 0$. Since $J$ is an ideal, $i \in J$. Hence $B(I, J) > 0$.

Proposition 18: Let $L$ be an ADFL and $(F_x(L), R)$ be a fuzzy Poset. For any $F, G \in F_x(L)$, $B(F, G) > 0$ if and only if for all $f \in F$ there exists $g \in G$ such that $A(g, f) > 0$. 


Proof: Let \( F, G \in F_L(L) \). Suppose \( B(F, G) > 0 \), and let \( f \in F \). Since \( F \subseteq G \), \( f \in G \). Then \( A(f, f) > 0 \). Conversely, suppose for all \( f \in F \) there exists \( g \in G \) such that \( A(g, f) > 0 \). Since \( G \) is an filter, \( f \in G \). Hence \( B(F, G) > 0 \).

Definition 19 ([7]): Let \((R, A)\) be a fuzzy poset. \((R, A)\) is a fuzzy lattice if and only if \( x \lor y \) and \( x \land y \) exist for all \( x, y \in R \).

Definition 20 ([7]): Let \((R, A)\) be a fuzzy lattice. \((R, A)\) is distributive if and only if \( x \land (y \lor z) = (x \land y) \lor (x \land z) \) and \( (x \lor y) \land (x \lor z) = x \lor (y \land z) \) for all \( x, y, z \in R \).

Theorem 21: Let \( L \) be an ADFL and \((I_L(L), B)\) be a fuzzy poset. Then \((I_L(L), B)\) forms a fuzzy lattice.

Proof: Let \( I \) and \( J \) are any arbitrary elements of \( I_L(L) \). By Theorem 9, both \( I \lor J \) and \( I \land J \) exist and they are in \((I_L(L), B)\).

First we show that \( I \lor J \) and \( I \land J \) are an upper bound and lower bound of the set containing \( I \) and \( J \) respectively.

**Proof of** \((I_L(L), B)\) **is the smallest upper bound of** \( I \) **and** \( J \). **Again, assume there exists an ideal** \( N \) **of** \( L \) **such that** \( I \land J \) **is the smallest upper bound of** \( I \) **and** \( J \) **respectively.**

**Proof of** \((I_L(L), B)\) **is the greatest lower bound of** \( I \) **and** \( J \). **To show** \((I_L(L), B)\) **is the greatest lower bound of** \( I \) **and** \( J \). **Again let** \((I_L(L), B)\) **be any non-empty subset of** \( I \). **Then** \((I_L(L), B)\) **forms a distributive fuzzy lattice.**

Corollary 23: Let \( P_L(L) \) be the set of all principal ideals of an ADFL \( L \), and \((P_L(L), B)\) be a fuzzy poset. Then \((P_L(L), B)\) forms a distributive fuzzy lattice.

Definition 25 ([111]): Let \((H, C)\) be a fuzzy lattice and \( H \) be any non-empty subset of \( H \). \( H \) is an ideal of \((H, C)\), if the following two conditions hold true.

\( (i) \) \( i_1, i_2 \in I \) implies \( i_1 \lor i_2 \in I \)

\( (ii) \) \( i \in I \) and \( x \in H \) such that \( C(x, i) > 0 \) implies \( x \in I \)

DFinition 26 ([111]): Let \((H, C)\) be a fuzzy lattice and \( F \) be any non-empty subset of \( H \). \( F \) is an ideal of \((H, C)\), if the following two conditions hold true.

\( (i) \) \( f_1, f_2 \in F \) implies \( f_1 \land f_2 \in F \)

\( (ii) \) \( f \in F \) and \( x \in H \) such that \( C(f, x) > 0 \) implies \( x \in F \)

From Corollary 23 and 28 we have seen that, \((P_L(L), B)\) and \((P_L(L), B)\) forms a fuzzy lattice, whenever both are fuzzy posets, and in the following two Lemmas, we define an ideal and filters of a fuzzy lattice \((P_L(L), B)\) and \((P_L(L), B)\) which are induced by an ideal \( I \) and a filter \( F \) of an ADFL \( L \).

We start by ideals and filters of a fuzzy lattice \((P_L(L), B)\), and in this paper \( I_L(L) \) denotes a distributive fuzzy lattice \((P_L(L), B)\) and \( L^d_L(L) \) denotes a distributive fuzzy lattice \((P_L(L), B)\). Consider two non-empty subset \( I \) and \( F \) of \( R \). Define \( I^d_1, I^d_2, F^d_1, F^d_2 \) by:

\( I^d_1 = \{i : i \in I\}, I^d_2 = \{i_1 : i_1 \in I\}, F^d_1 = \{f_1 : f_1 \in F\}, \) and \( F^d_2 = \{f_2 : f_2 \in F\} \). It is clear that \( I^d_1 \) and \( F^d_2 \) are subsets of \((P_L(L), B)\) and \( I^d_2 \) and \( F^d_1 \) are subsets of \((P_L(L), B)\).

We have the following Theorem.

Theorem 27: Let \( I \) be any non-empty subset of \( R \), and \( L \) be an ADFL. Then the followings are equivalent:

\( (i) \) \( i_1, i_2 \in I \) implies \( i_1 \lor i_2 \in I \)

\( (ii) \) \( i \in I \) and \( x \in H \) such that \( C(x, i) > 0 \) implies \( x \in I \)

\( (iii) \) \( f_1, f_2 \in F \) implies \( f_1 \land f_2 \in F \)

\( (iv) \) \( f \in F \) and \( x \in H \) such that \( C(f, x) > 0 \) implies \( x \in F \)
1. \( I \) is an ideal of \( L \);  
2. \( I_A^1 \) is an ideal of a distributive fuzzy lattice \( L_B^1 \);  
3. \( I_A^1 \) is a filter of a distributive fuzzy lattice \( L_B^1 \).  

**Proof:** (1) \( \Rightarrow \) (2) Suppose \( I \) is an ideal of \( L \). We need to show that \( I_A^1 \) is an ideal of \( L_B^1 \).  
(a) Let \( [i_1]_A \) and \( [i_2]_A \) be any two arbitrary elements of \( I_A^1 \). Then \([i_1]_A \wedge [i_2]_A = [i_1 \lor i_2]_A \). Since \( i_1 \lor i_2 \in I \), \( [i_1]_A \lor [i_2]_A = ([i_1 \lor i_2]_A \lor [i_1]_A) \).  
(b) Let \( [i]_A \in I_A^1 \) and \( (x)_A \in P(A)(L) \) such that \( B((x)_A, [i]_A) > 0 \). Then there exists \( y \in [i]_A \) such that \( A((x) \land y) > 0 \), in particular \( A((x) \land y) \). Since \( I \) is an ideal of \( L \) and \( i \land x \in I \), and then \( (x)_A \in I_A^1 \). Therefore, \( I_A^1 \) is an ideal of \( L_B^1 \).  

(2) \( \Rightarrow \) (3) Suppose \( I_A^1 \) is an ideal of \( L_B^1 \). We need to show that \( I_A^1 \) is a filter of \( I_A^1 \).  
(a) Let \( [i_1]_A \) and \( [i_2]_A \) be any two arbitrary elements of \( I_A^1 \). Then \([i_1]_A \wedge [i_2]_A = [i_1 \lor i_2]_A \). Since \( I \) is an ideal of \( L \), \( i_1 \lor i_2 \in I \). Therefore, \( I_A^1 \) is a filter of \( L_B^1 \).  
(b) Let \( [i]_A \in I_A^1 \) and \( (x)_A \in P(A)(L) \) such that \( B((i)_A, (x)_A) > 0 \). Then \( A((x) \land i) > 0 \) or \( A((x) \land i) > 0 \). Hence, \( x \land i \in I \). Therefore, \( I_A^1 \) is an ideal of \( L_B^1 \).  

**Theorem 28:** Let \( F \) be any non empty subset of \( R \), and \( L \) be an ADFL. Then the followings are equivalent.  
1. \( F \) is a filter of \( L \);  
2. \( F_A^1 \) is a filter of a distributive fuzzy lattice \( L_B^1 \);  
3. \( F_A^1 \) is an ideal of a distributive fuzzy lattice \( L_B^1 \).  

**Proof:** (1) \( \Rightarrow \) (2) Suppose \( F \) is a filter of \( L \). Then we need to show that \( F_A^1 \) is a filter of \( L_B^1 \).  
(a) Let \( (f_1)_A \) and \( (f_2)_A \) are any two arbitrary elements of \( F_A^1 \). Then \( (f_1)_A \wedge (f_2)_A = (f_1 \land f_2)_A \in F_A^1 \). Since \( F \) is a filter and \( f_1 \land f_2 \in F \).  
(b) Let \( (f)_A \in F_A^1 \) and \( (x)_A \in P(A)(L) \) such that \( B((x)_A, (f)_A) > 0 \). Then \( f \in (x)_A \) and it implies that \( A((f \land x) \wedge f) > 0 \) or \( A((f \land x) \wedge f) > 0 \). Hence, \( x \in F \) and \( x \in F \). Therefore, \( F_A^1 \) is a filter of a distributive fuzzy lattice \( L_B^1 \).  

(2) \( \Rightarrow \) (3) Suppose \( F_A^1 \) is a filter of \( L_B^1 \). Then we need to show that \( F_A^1 \) is an ideal of \( L_B^1 \).  
(a) Let \( (f_1)_A \) and \( (f_2)_A \) are any two arbitrary elements of \( F_A^1 \). Then \( (f_1)_A \wedge (f_2)_A = (f_1 \land f_2)_A \in F_A^1 \). Since \( F \) is a filter and \( f_1 \land f_2 \in F \).  
(b) Let \( (f)_A \in F_A^1 \) and \( (x)_A \in P(A)(L) \) such that \( B((x)_A, (f)_A) > 0 \). Then \( x \in (f)_A \) and it implies that \( A((f \land x) \wedge f) > 0 \) or \( A((f \land x) \wedge f) > 0 \). Since \( f \in F \) and \( F \) is a filter, \( x \in F \) and it implies that \( (x)_A \in F_A^1 \). Therefore, \( F_A^1 \) is an ideal of a distributive fuzzy lattice \( L_B^1 \).  

(3) \( \Rightarrow \) (1) Suppose \( F_A^1 \) is an ideal of a distributive fuzzy lattice \( L_B^1 \). We need to show that \( F \) is a filter of \( L \).  
(a) Let \( f_1, f_2 \in F \). Then \( (f_1)_A \wedge (f_2)_A \in F_A^1 \). Since \( F_A^1 \) is an ideal of \( L_B^1 \), \( (f_1)_A \wedge (f_2)_A \in F_A^1 \). Hence \( f_1 \land f_2 \in F \).