Stability Analysis of Traveling Waves to Advection-Diffusion Equation Involving Square-root

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***Abstract*—In this paper, we study the existence and stability of advection-diffusion equation involving square-root. We first change the original equation into the traveling wave by using ansatz transformation. Then, we apply the appropriate perturbation to establish the energy estimate under small perturbation and large wave amplitude. These results of energy estimates are used to prove the stability of traveling wave solutions.**

***Index Terms*—Stability, large wave amplitude, small perturbations.**

I. Introduction

We consider the following advection-diffusion equation,

$$m_t + (\sqrt{m})_x = m_{xx}, \tag{1}$$

where $m = m(x,t)$ and the initial state

$$m(x,0) = m_0(x) \to m_\pm \text{ as } x \to \pm\infty. \tag{2}$$

The equation (1) is the special case of the following equation,

$$m_t + (g(m))_x = \alpha m_{xx}, \tag{3}$$

where this equation was studied by Il'in and Oleinik [1] and Sattinger [2] for the maximum principle and spectral analysis respectively of shock waves, for $\alpha > 0$ and a smooth function $g(m)$. Mickens and Oyedeji [3] studied the traveling waves to Burger's and non-diffusion Fisher equation. These two equations involving square-root were also the the special case of (3).

$$\begin{aligned} u_t + a_1\sqrt{u}u_x &= D_1 u_{xx} \\ u_t + a_2\sqrt{u}u_x &= \lambda_1\sqrt{u} - \lambda_2 u \end{aligned} \tag{4}$$

where $a_1, a_2 > 0, D_1 > 0, \lambda_1, \lambda_2 > 0$.

Other studies related to square-root $\sqrt{u}$ were established by Buckmire et. al [4], Jordan [5], and Mickens [6], [7]. This current paper, we focus on the existence and stability of traveling waves to advection-diffusion equation (1). Moreover, we employ the energy estimates under small perturbation and arbitrary wave amplitude. This technique was also studied in [8], [9] for chemotaxis model. Hu [10] employed the energy method to deal with the stability of traveling waves to coupled Burger's equation,

$$\begin{aligned} u_t + \left(\frac{1}{2}u^2 + \frac{1}{2}b^2\right)_x &= \mu u_{xx}, \\ b_t + (ub)_x &= \nu b_{xx}, \end{aligned} \tag{5}$$

where the small coefficient was not required.

A similar problem to (5) was studied by Li and Wang [8] as shown the following system of equation,

$$\begin{aligned} u_t - (uv)_x &= D u_{xx}, \\ v_t + (\varepsilon v^2 - u)_x &= \varepsilon v_{xx}, \end{aligned} \tag{6}$$

where the smallness of wave amplitude and coefficients were required in (6). The main problem of this paper is the square root which is a challenge to study. We also state the proof of existence and stability for traveling wave solutions of (1) with $m_+ > 0$. Then this paper is organized as follows. In Section II, we transform the original equation (1) into the traveling waves by applying the ansatz transformation, and derive the appropriate perturbations which implies L^2 distance between the solution m and M of the equation (1). Section III presents the energy estimate of transformed problem. In Section IV, we finally prove the stability of traveling waves to equation (1).

II. Transformation of the Problem

We first substitute the following ansatz transformation

$$m(x,t) = M(\zeta),\ \zeta = x - st \tag{7}$$

into (1). Moreover, ζ and s denote moving frame variable and wave speed respectively. Then, the traveling waves M satisfy

$$-sM_\zeta + (\sqrt{M})_\zeta = M_{\zeta\zeta} \tag{8}$$

with the following boundary conditions

$$M(\zeta) \to m_\pm \text{ as } \zeta \to \pm\infty. \tag{9}$$

We further integrate (8) with respect to ζ

$$-sM + sm_\pm + \sqrt{M} - \sqrt{m_\pm} = M_\zeta, \tag{10}$$

and apply the fact $M_\zeta \to 0$ as $\zeta \to \pm\infty$, then one has the following Rankine-Hugoniot condition

$$s(m_- - m_+) = \sqrt{m_-} - \sqrt{m_+} \tag{11}$$

which gives the wave speed

$$s = \frac{\sqrt{m_-} - \sqrt{m_+}}{m_- - m_+} > 0. \tag{12}$$

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We further, present the following proposition to deal with the existence of traveling wave (1).

Lemma 1: Assume that $m_\pm$ satisfy (11). Then a monotone traveling wave $M(x-st)$ to (8) exists, which is unique up to a translation and holds $M_\zeta < 0$. Moreover, M decays exponentially fast with rates

$$M - m_\pm \sim e^{\sigma_\pm z} \text{ as } z \to \pm\infty,$$

where

$$\sigma_\pm = \frac{1-2s\sqrt{m_\pm}}{2\sqrt{m_\pm}},$$

and the wave speed s is given in (12).
We define the following perturbation for the transformed equation (1).

$$\varphi_0(\zeta) = \int_{-\infty}^{\zeta} (m_0 - M)(y)dy,$$

which is the zero mass perturbation (see [11], [12]). Then we have the following stability results.

Theorem 1: Consider the traveling wave solution $M(x-st)$ given in Lemma 1. If $\|m_0 - M\|_1 + \|\phi_0\| \le \varepsilon_0$ for a constant $\varepsilon_0 > 0$, then there exists unique global solution $m(x,t)$ to (1)-(2), satisfying

$$m - M \in C([0,\infty); H^1) \cap L^2([0,\infty); H^1),$$

and

$$\sup_{x\in\mathbb{R}} |m(x,t) - M(x-st)| \to 0 \quad \text{as} \quad t \to +\infty.$$

We change the variables $(x,t) \to (\zeta = x - st, t)$ in (1) to get

$$m_t - sm_\zeta + (\sqrt{m})_\zeta = (m)_{\zeta\zeta} \tag{13}$$

We decompose the solution of (13) as

$$m(\zeta,t) = M(\zeta) + \varphi_\zeta(\zeta,t). \tag{14}$$

Then

$$\varphi(\zeta,t) = \int_{-\infty}^{\zeta} (m(y,t) - M(y))dy \tag{15}$$

We substitute (14) into (13) and integrate the results in ζ to get

$$\varphi_t = s\varphi_\zeta - \sqrt{\varphi_\zeta + M} + \varphi_{\zeta\zeta}, \tag{16}$$

where the initial data of φ is given

$$\varphi(\zeta,0) = \varphi_0(\zeta) = \int_{-\infty}^{\zeta} (m_0 - M)dy, \tag{17}$$

for $\varphi_0(\pm\infty) = 0$. We further find the solution of reformulated problem (16)-(17) in the space

$$X(0,T) := \left\{\varphi(\zeta,t) \in C([0,T), H^2) : \varphi_\zeta \in L^2((0,T); H^2))\right\}$$

for $0 < T \le +\infty$. Let

$$N(t) := \sup_{0\le\tau\le t} \left\{\|\varphi(.,\tau)\|_2\right\}.$$

From the Sobolev inequality $\|f\|_{L^\infty} \le \sqrt{2}\|f\|_{L^2}^{\frac{1}{2}}\|f_x\|_{L^2}^{\frac{1}{2}}$, it holds that

$$\sup_{\tau\in[0,t]} \{\|\varphi(\cdot,\tau)\|_{L^\infty}, \|\varphi_\zeta(\cdot,\tau)\|_{L^\infty}\} \le N(t).$$

For (16)-(17), we have the following global well-posedness.

Theorem 2: Under the assumptions in Theorem 1. If $N(0) \le \delta_1$ for a constant $\delta_1 > 0$, then there exists a unique global solution $\varphi \in X(0,+\infty)$(16)-(17), satisfying

$$\|\varphi(.,t)\|_2^2 + \int_0^t \|\varphi_\zeta(.,\tau)\|_2^2 \, d\tau \le C\|\varphi_0\|_2^2. \tag{18}$$

Moreover, it holds that

$$\sup_{\zeta\in R} |\varphi_\zeta(\zeta,t)| \to 0 \text{ as } t \to +\infty \tag{19}$$

We refer to [13] for the local existence proof, then we only need to establish the following a priori estimate.

Proposition 1: Let $\varphi \in X(0,T)$ be solution of (16)-(17) for particular time $T > 0$. If $N(T) < \varepsilon_1$ for a constant $\varepsilon_1 > 0$ which is independent of T, then φ satisfies (18) for any $0 \le t \le T$.

III. Energy estimates

Now, we are ready to establish the a priori estimates of φ of (16)-(17), and hence prove Proposition 1. As the first step, we prove L^2 estimate.

Lemma 2: Under assumptions in Theorem 1. For a constant $C > 0$, one has

$$\|\varphi(.,t)\|^2 + \int_0^t \|\varphi_\zeta(.,\tau)\|^2 d\tau \le C\|\varphi_0\|^2 + C\int_0^t \int \varphi^2 \tag{20}$$

Proof: Multiplying (16) by φ/M and integrating the results to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \frac{\varphi^2}{M} + \int \frac{\varphi_\zeta^2}{M} \\ &= \int \frac{\varphi\varphi_\zeta M_\zeta}{M^2} + \int \frac{s\varphi\varphi_\zeta}{M} - \int \frac{\varphi\sqrt{\varphi_\zeta + M}}{M}.\end{aligned} \tag{21}$$

By applying the following inequality

$$\begin{aligned}\sqrt{\varphi_\zeta + M} &\le \sqrt{\varphi_\zeta^2 + \varphi_\zeta M + M^2} \\ &= \sqrt{(\varphi_\zeta + M)^2} = \varphi_\zeta + M,\end{aligned} \tag{22}$$

into (21), one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \frac{\varphi^2}{M} + \int \frac{\varphi_\zeta^2}{M} \\ &\le -\int \varphi\varphi_\zeta \left(\frac{1}{M}\right)_\zeta + \int \frac{s\varphi\varphi_\zeta}{M} - \int \frac{\varphi(\varphi_\zeta + M)}{M}.\end{aligned} \tag{23}$$

Noting that

$$\begin{aligned}&\int \left(-\varphi\varphi_\zeta \left(\frac{1}{M}\right)_\zeta + \frac{s\varphi\varphi_\zeta}{M} - \frac{\varphi(\varphi_\zeta + M)}{M}\right) \\ &= \int -\frac{\varphi^2}{2}\left(-\left(\frac{1}{M}\right)_{\zeta\zeta} + \left(\frac{s}{M} - \frac{1}{M}\right)_\zeta\right)\end{aligned}$$

From (10) and $M_\zeta < 0$, we get

$$\begin{aligned}
&\left(\frac{1}{M}\right)_{\zeta\zeta} - \left(\frac{s}{M} - \frac{1}{M}\right)_\zeta \\
&= \left(\left(\frac{1}{M}\right)_\zeta - \left(\frac{s}{M} - \frac{1}{M}\right)\right)_\zeta \\
&= \left(1 + \frac{3\sqrt{M}}{2} + \frac{2(sm_+ - \sqrt{m_+})}{M}\right) M_\zeta \\
&< 0
\end{aligned} \tag{24}$$

Substituting (24) into (23), using the fact that $M \geq m_+ > 0$, then we complete the proof of Lemma 2. ■

By the similar way with L^2 estimate, the second step is to present H^1 estimate of φ.

Lemma 3: Under assumptions in Theorem 1. For a constant $C > 0$, one has

$$\|\varphi(.,t)\|_1^2 + \int_0^t \|\varphi_\zeta(.,\tau)\|_1^2 d\tau \leq C\|\varphi_0\|_1^2 + C\int_0^t \int \varphi^2 \tag{25}$$

Proof: Differentiating (16) in z gives

$$\varphi_{\zeta t} = s\varphi_{\zeta\zeta} - \frac{\varphi_{\zeta\zeta} + M_\zeta}{2\sqrt{\varphi_\zeta + M}} + \varphi_{\zeta\zeta\zeta}$$

Since $1/\sqrt{\varphi_\zeta + M} = \sqrt{\varphi_\zeta + M}/(\varphi_\zeta + M) \leq \varphi_\zeta + M/(\varphi_\zeta + M) = 1$, then the above equation becomes

$$\varphi_{\zeta t} \leq s\varphi_{\zeta\zeta} + \frac{\varphi_{\zeta\zeta} + M_\zeta}{2} + \varphi_{\zeta\zeta\zeta} \tag{26}$$

Multiplying (26) by φ_ζ/M and integrating the results, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{\varphi_\zeta^2}{M} + \int \frac{\varphi_{\zeta\zeta}^2}{M} \\
&\leq \int \frac{\varphi_\zeta \varphi_{\zeta\zeta} M_z}{M^2} + \int \frac{s\varphi_\zeta\varphi_{\zeta\zeta}}{M} + \int \frac{\varphi_\zeta\varphi_{\zeta\zeta} + \varphi_\zeta M_\zeta}{2M}
\end{aligned} \tag{27}$$

Noting that

$$\begin{aligned}
\int \frac{\varphi_\zeta\varphi_{\zeta\zeta}M_\zeta}{M^2} &= -\int \varphi_\zeta\varphi_{\zeta\zeta}\left(\frac{1}{M}\right)_\zeta = \int \frac{\varphi_\zeta^2}{2}\left(\frac{1}{M}\right)_{\zeta\zeta} \\
\int \frac{s\varphi_\zeta\varphi_{\zeta\zeta}}{M} &= -s\int \frac{\varphi_\zeta^2}{2}\left(\frac{1}{M}\right)_\zeta \\
\int \frac{\varphi_\zeta\varphi_{\zeta\zeta}}{M} &= -\int \frac{\varphi_\zeta^2}{2}\left(\frac{1}{M}\right)_\zeta
\end{aligned} \tag{28}$$

By employing (28) into (27), then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{\varphi_\zeta^2}{M} + \int \frac{\varphi_{\zeta\zeta}^2}{M} \\
&\leq \int -\frac{\varphi_\zeta^2}{2}\left(-\left(\frac{1}{M}\right)_{\zeta\zeta} + s\left(\frac{1}{M}\right)_\zeta - \left(\frac{1}{M}\right)_\zeta\right) \\
&\quad - \int \frac{\varphi_\zeta}{2}M\left(\frac{1}{M}\right)_\zeta
\end{aligned} \tag{29}$$

We combine (29) with the results in (20). Then, we apply (10) and (24) to get

$$\int \frac{\varphi_\zeta^2}{M} + \int_0^t \int \frac{\varphi_{\zeta\zeta}^2}{M} \leq \int \frac{\varphi_{0\zeta}}{M} + C\int_0^t \int \varphi^2$$

We use the fact $M \geq m_+ > 0$ to above inequality, then the proof (25) is completed. ■

By the similar way with L^2 and H^1 estimates, the third step is to present H^2 estimate of φ.

Lemma 4: Under assumptions in Theorem 1. For a constant $C > 0$, one has

$$\|\varphi(.,t)\|_2^2 + \int_0^t \|\varphi_\zeta(.,\tau)\|_2^2 \leq C\|\varphi_0\|_2^2 + C\int_0^t \int \varphi^2. \tag{30}$$

Proof: Differentiating (26) with respect to z gives

$$\varphi_{\zeta\zeta t} \leq s\varphi_{\zeta\zeta\zeta} + \frac{\varphi_{\zeta\zeta\zeta} + M_{\zeta\zeta}}{2} + \varphi_{\zeta\zeta\zeta\zeta} \tag{31}$$

Multiplying (31) by $\varphi_{\zeta\zeta}/M$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int \frac{\varphi_{\zeta\zeta}^2}{M} + \int \frac{\varphi_{\zeta\zeta\zeta}^2}{M} \leq \int \frac{\varphi_{\zeta\zeta}\varphi_{\zeta\zeta\zeta}M_\zeta}{M^2} + \int \frac{s\varphi_{\zeta\zeta}\varphi_{\zeta\zeta\zeta}}{M} \\
+ \int \frac{\varphi_{\zeta\zeta}\varphi_{\zeta\zeta\zeta} + \varphi_{\zeta\zeta}M_{\zeta\zeta}}{2M}
\end{aligned} \tag{32}$$

Noting that

$$\begin{aligned}
\int \frac{\varphi_{\zeta\zeta}\varphi_{\zeta\zeta\zeta}M_\zeta}{M^2} &= -\int \varphi_{\zeta\zeta}\varphi_{\zeta\zeta\zeta}\left(\frac{1}{M}\right)_\zeta = \int \frac{\varphi_{\zeta\zeta}^2}{2}\left(\frac{1}{M}\right)_{\zeta\zeta} \\
\int \frac{s\varphi_{\zeta\zeta}\varphi_{\zeta\zeta\zeta}}{M} &= -s\int \frac{\varphi_{\zeta\zeta}^2}{2}\left(\frac{1}{M}\right)_\zeta \\
\int \frac{\varphi_{\zeta\zeta}\varphi_{\zeta\zeta\zeta}}{M} &= -\int \frac{\varphi_{\zeta\zeta}^2}{2}\left(\frac{1}{M}\right)_\zeta
\end{aligned} \tag{33}$$

Substituting (33) into (32), then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{\varphi_{\zeta\zeta}^2}{M} + \int \frac{\varphi_{\zeta\zeta\zeta}^2}{M} \\
&\leq \int -\frac{\varphi_{\zeta\zeta}^2}{2}\left(-\left(\frac{1}{M}\right)_{\zeta\zeta} + s\left(\frac{1}{M}\right)_\zeta - \left(\frac{1}{M}\right)_\zeta\right) \\
&\quad - \int \frac{\varphi_{\zeta\zeta}}{2}M\left(\frac{1}{M}\right)_\zeta
\end{aligned} \tag{34}$$

By employing (10), (24), and the fact $M \geq m_+ > 0$, then Lemma 4 is proved. ■

IV. Proof of Theorem 2

Proof: Proposition 1 follows from Lemma 2 to Lemma 4. Now, we are ready to prove the main results by the transformation (14). Theorem 1 is a consequence of Theorem 2. The a priori estimate (18) guarantees that $N(t)$ is small if $N(0)$ is small enough. Thus, applying the standard extension procedure, we get the global well-posedness of (16)-(17) in $X(0, +\infty)$.

We further prove the convergence (19). By the global estimate (18), one has

$$\int_0^t \int_{-\infty}^{\infty} \varphi_\zeta^2(\zeta,\tau)d\zeta d\tau \leq C\|\varphi_0\|_2^2 < \infty \tag{35}$$

It follows from the first equation of (16) and Young's inequality,

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}\varphi_\zeta^2(\zeta,t)d\zeta &= -2\int_{-\infty}^{\infty}\varphi_t\varphi_{\zeta\zeta}d\zeta\\ &= -2\int_{-\infty}^{\infty}\varphi_{\zeta\zeta}(s\varphi_\zeta-\sqrt{\varphi_\zeta+M}+\varphi_{\zeta\zeta})\\ &\leq C\int_{-\infty}^{\infty}(\varphi_{\zeta\zeta}^2+\varphi_\zeta^2).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\int_0^\infty\left|\frac{d}{dt}\int_{-\infty}^{\infty}\varphi_\zeta^2(\zeta,t)d\zeta\right|\\ &\leq C\int_0^\infty\int_{-\infty}^{\infty}(\varphi_{\zeta\zeta}^2+\varphi_\zeta^2)\leq C\|\varphi_0\|_2^2<\infty.\end{aligned}\tag{36}$$

From (35) and (36), obtained

$$\int_{-\infty}^{\infty}\varphi_\zeta^2(\zeta,t)d\zeta\to 0 \text{ as } t\to+\infty.$$

By Cauchy-Schwarz inequality, we further have

$$\begin{aligned}\varphi_\zeta^2(\zeta,t) &= 2\int_{-\infty}^{\zeta}\varphi_\zeta\varphi_{\zeta\zeta}(y,t)dy\\ &\leq 2\left(\int_{-\infty}^{+\infty}\varphi_\zeta^2(y,t)dy\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\varphi_{\zeta\zeta}^2(y,t)dy\right)^{\frac{1}{2}}\\ &\to 0 \text{ as } t\to+\infty\end{aligned}$$

Hence (19) is established which implies that the Theorem 2 is finally proved. ∎

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