Optimal Feeding Strategy on Microalgae Growth in Fed-Batch Bioreactor Model

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***Abstract*—Some countries in the world turn to alternative energy source to fulfill their necessity of fuel. One of the alternative fuels is biodiesel. Raw material of biodiesel can be produced by microalgae cultivation in fed-batch bioreactor. To improve the productivity of microalgae cultivation, we need to determine the optimal control of microalgae growth. This paper discusses mathematical model of microalgae growth in fed-batch bioreactor, and solves the optimal feeding strategy problem by using Pontryagin Minimum Principle. Then we compare the controlled microalgae growth model with the uncontrolled one. Numerical simulation with DOTcvpSB shows that the controlled microalgae growth model yields more harvest and less cost function than the uncontrolled one.**

***Index Terms*—Bioreactor model, optimal control.**

I. Introduction

BIODESEL is one of the alternative energy sources that becomes popular in recent decades. Biodesel can be produced from microalgae biomass. Microalgae as one of biomass sources, has many advantages, such as can be cultivated throughout the year, need less water and field. Microalgae can grow fast in rich oil content. Oil content of microalgae can reach 80% of its dry biomass weight. Oil content that produced from microalgae cultivation depends on microalgae growth rate and oil content in its biomass [1].

De la Hoz Siegler et al. [2] constructed a mathematical model to describe microalgae growth in fed-batch bioreactor. Then the optimization of the model to find the optimal feeding strategy is solved by using adaptive model predictive control [3]. The optimal control problem of the model is also discussed by Abdollahi and Dubljevic [4]. They solved the optimal control problem to optimize biomass and lipid productivity by using Interior Point Optimization (IPOPT), Model Predictive Control (MPC), and Moving Horizon Estimation (MHE).

The model constructed in [2] assumes glycine, glucose, biomass, lipid, nitrogen concentration, and bioreactor volume as state variables. In this paper, we discuss a modification of the model and then solve the optimal control problem to optimize biomass and lipid productivity and microalgae feeding cost by using Pontryagin Minimum Principle.

II. Mathematical Model of Microalgae Growth in Fed-Batch Bioreactor

The dynamic of microalgae growth in fed-batch bioreactor is modeled by De la Hoz Siegler et al. [2] by (1)-(6). System (1)-(6) describes biomass and lipid production of microalgae Auxenochlorella protothecoides. Microalgae growth in bioreactor needs some substrates as nutrition sources. In this model, De la Hoz Siegler et al. use two substrates, i.e. glycine as nitrogen source, and glucose as carbon source. The model assumes that the microalgae cell contains three components, i.e. active biomass, lipid, and nitrogen. These three components can convert from one component to another component with constant yields $Y_{i/j}$.

$$\frac{dS_1}{dt} = -\rho x + s_1^i \frac{f_1^i}{V} - S_1 D \tag{1}$$

$$\frac{dS_2}{dt} = \frac{-1}{Y_{x/s}}\mu x - \frac{1}{Y_{p/s}}\pi x - k_m x + s_2^i \frac{f_2^i}{V} - S_2 D \tag{2}$$

$$\frac{dx}{dt} = \mu x - xD \tag{3}$$

$$\frac{dp}{dt} = \pi x - \frac{1}{Y_{x/p}}\mu x - pD \tag{4}$$

$$\frac{dq}{dt} = \rho x - \frac{1}{Y_{x/q}}\mu x - qD \tag{5}$$

$$\frac{dV}{dt} = VD - f^o \tag{6}$$

The model consider six state variables, they are S_1, S_2, x, p, q, V that represent glycine, glucose, biomass, lipid, and nitrogen concentration in fed-batch bioreactor, and volume of the bioreactor, respectively. ρ is the nitrogen source uptake rate, π is the lipid production rate, and μ is the microalgae growth rate. s_1^i and s_2^i are the concentration of feeding nutrition. f_1^i and f_2^i are the feeding nutrition rate. f^o is the outflow, k_m is the maintenance factor, and $D = (f_1^i + f_2^i)/V$ is the dilution rate.

III. Methods

This section discusses the methods used in this study. Firstly, we modify the model by using different assumption of variable state and parameter. Then we discuss the optimal control problem and solve it by using Pontryagin Minimum Principle and DOTcvpSB.

A. Modification of The Mathematical Model

In this study, the volume of fed-batch bioreactor is assumed as a constant parameter. So, we only consider S_1, S_2, x, p, q as the state variables of the model. We also assume the nitrogen source uptake rate, the lipid production rate, and the microalgae growth rate as (7)-(9). Where ρ_{max} is the maximal nitrogen source uptake rate, π_{max} is the maximal

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lipid production rate, μ_{max} is the maximal microalgae growth rate, and

$$\mu = \mu_{max} \frac{S_2}{K_{S_2} + S_2} \frac{q}{K_q + q} \tag{7}$$

$$\pi = \pi_{max} \frac{S_2}{K_{S_2} + S_2} \tag{8}$$

$$\rho = \rho_{max} \frac{S_1}{K_{S_1} + S_1} \tag{9}$$

K_{S_1}, K_{S_2}, K_q are the half saturation constant of glycine, glucose, and nitrogen concentration, respectively.

B. Pontryagin Minimum Principle

Optimal control problem consists of three main components, i.e. mathematical model, objective function, and boundary conditions and physical constrains of state/control variables. Let the mathematical model be given by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t),$$

and objective function

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \phi(\mathbf{x}(t), \mathbf{u}(t), t) dt,$$

with boundary conditions

$$\mathbf{x}(t_0 = 0) = \mathbf{x}_0 \quad \text{and} \quad \mathbf{x}(t_f) = \mathbf{x}_f,$$

and physical constraints

$$\mathbf{U}_- \leq \mathbf{u}(t) \leq \mathbf{U}_+ \quad \text{and} \quad \mathbf{X}_- \leq \mathbf{x}(t) \leq \mathbf{X}_+.$$

To solve the optimal control problem with Pontryagin Minimum Principle, we need to do the steps as follows [5]:

1) Form the Hamiltonian function.
$$H(\mathbf{x}, \mathbf{u}, \lambda(t), t) = \phi(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$
2) Find the optimal control $\mathbf{u}^*(t) = \mathbf{h}(\mathbf{x}^*(t), \lambda^*(t), t)$ by minimizing H w.r.t. $\mathbf{u}(t)$.
$$\left(\frac{\partial H}{\partial \mathbf{u}}\right)_* = 0$$
3) Substitute the optimal control to Hamiltonian function.
$$H^*(\mathbf{x}^*(t), \mathbf{h}(\mathbf{x}^*(t), \lambda^*(t), t), \lambda^*(t), t)$$
4) Solve the state and costate differential equations
$$\dot{\mathbf{x}}^*(t) = \left(\frac{\partial H}{\partial \lambda}\right)_* \quad \text{and} \quad \dot{\lambda}^*(t) = -\left(\frac{\partial H}{\partial \mathbf{x}}\right)_*$$
with initial and final conditions
$$\left(H + \frac{\partial S}{\partial t}\right)_{*t_f} \delta t_f + \left(\left(\frac{\partial S}{\partial \mathbf{x}}\right) - \lambda(t)\right)_{*t_f} \delta \mathbf{x}_f = 0.$$

C. DOTcvpSB

DOTcvpSB is a toolbox of MATLAB that can be used to solve an optimal control problem numerically. DOTcvpSB focuses on system biology problems. To solve an optimal control problem with this toolbox, we need to define the ordinary differential equations that describe the system, the objective function, initial and terminal time problems, constraints of the variables, initial and terminal conditions. This toolbox also allows us to check the simulation of the systems, which only shows the dynamics of the systems without optimization [6].

IV. Discussion and Results

A. Mathematical Model

Consider the assumption explanation in the method section, we can express the mathematical model of microalgae growth in fed-batch bioraector as (10)-(14).

$$\frac{dS_1}{dt} = -\rho_{max} \frac{S_1}{K_{S_1} + S_1} x + s_1^i \frac{f_1^i}{V} - S_1 D, \tag{10}$$

$$\frac{dS_2}{dt} = \left(\frac{-1}{Y_{x/s}} \mu_{max} \frac{q}{K_q + q} - \frac{1}{Y_{p/s}} \pi_{max}\right) \frac{S_2}{K_{S_2} + S_2} x$$
$$- k_m x + s_2^i \frac{f_2^i}{V} - S_2 D, \tag{11}$$

$$\frac{dx}{dt} = \mu_{max} \frac{S_2}{K_{S_2} + S_2} \frac{q}{K_q + q} x - xD, \tag{12}$$

$$\frac{dp}{dt} = \pi_{max} \frac{S_2}{K_{S_2} + S_2} x - \frac{1}{Y_{x/p}} \mu_{max} \frac{S_2}{K_{S_2} + S_2} \frac{q}{K_q + q} x$$
$$- pD, \tag{13}$$

$$\frac{dq}{dt} = \rho_{max} \frac{S_1}{K_{S_1} + S_1} x - \frac{1}{Y_{x/q}} \mu_{max} \frac{S_2}{K_{S_2} + S_2} \frac{q}{K_q + q} x$$
$$- qD. \tag{14}$$

B. Optimal Control Problem

The mathematical model in this optimal control problem is defined by (10)-(14), where f_1^i and f_2^i are defined as control variables. The aim of this optimal control problem is to maximize biomass and lipid concentration and minimize the feeding cost. Thus, the objective function of the problem can be written as

$$J = \int_{t_0}^{t_f} \left(\frac{C_1}{2} u_1^2 + \frac{C_2}{2} u_2^2 - x - p\right) dt. \tag{15}$$

The constraint of control variables are $0 \leq u_1 \leq 2$ and $0 \leq u_2 \leq 10$. We have applied the Pontryagin Minimum Principle to the above optimal control problem, as follows.

1) Hamiltonian function

$$\begin{aligned}
H = & -x - p + \frac{C_1}{2} u_1^2 + \frac{C_2}{2} u_2^2 \\
& + \lambda_1 \left(-\rho_{max} \frac{S_1}{K_{S_1} + S_1} x + s_1^i \frac{u_1}{V} - S_1 \frac{u_1 + u_2}{V}\right) \\
& + \lambda_2 \left(\frac{-1}{Y_{x/s}} \mu_{max} \frac{S_2}{K_{S_2} + S_2} \frac{q}{K_q + q} x \right. \\
& \left. \quad - \frac{1}{Y_{p/s}} \pi_{max} \frac{S_2}{K_{S_2} + S_2} x\right) \\
& + \lambda_2 \left(-k_m x + s_2^i \frac{u_2}{V} - S_2 \frac{u_1 + u_2}{V}\right) \\
& + \lambda_3 \left(\mu_{max} \frac{S_2}{K_{S_2} + S_2} \frac{q}{K_q + q} x - x \frac{u_1 + u_2}{V}\right) \\
& + \lambda_4 \left(\pi_{max} \frac{S_2}{K_{S_2} + S_2} x \right. \\
& \left. \quad - \frac{1}{Y_{x/p}} \mu_{max} \frac{S_2}{K_{S_2} + S_2} \frac{q}{K_q + q} x - p \frac{u_1 + u_2}{V}\right) \\
& + \lambda_5 \left(\rho_{max} \frac{S_1}{K_{S_1} + S_1} x \right.
\end{aligned}$$

$$-\frac{1}{Y_{x/q}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}x - q\frac{u_1+u_2}{V}\Big)$$

2) Optimal control law

$$u_1^* = \frac{1}{C_1 V}\left(-\lambda_1 s_1^i + \lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 x + \lambda_4 p + \lambda_5 q\right)$$

$$u_2^* = \frac{1}{C_2 V}\left(-\lambda_2 s_2^i + \lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 x + \lambda_4 p + \lambda_5 q\right)$$

3) Optimal Hamiltonian function

$$\begin{aligned}
H^* &= -x - p - \lambda_1\left(\rho_{max}\frac{S_1}{K_{S_1}+S_1}x\right) \\
&+ \lambda_2\left(\frac{-1}{Y_{x/s}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}x - \frac{1}{Y_{p/s}}\pi_{max}\frac{S_2}{K_{S_2}+S_2}x - k_m x\right) \\
&+ \lambda_3\left(\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}x\right) \\
&+ \lambda_4\left(\pi_{max}\frac{S_2}{K_{S_2}+S_2}x - \frac{1}{Y_{x/p}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}x\right) \\
&+ \lambda_5\left(\rho_{max}\frac{S_1}{K_{S_1}+S_1}x - \frac{1}{Y_{x/q}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}x\right) \\
&- \frac{\lambda_1^2(s_1^i)^2}{2C_1V^2} - \frac{\lambda_1^2S_1^2}{2C_1V^2} - \frac{\lambda_2^2S_2^2}{2C_1V^2} - \frac{\lambda_3^2x^2}{2C_1V^2} \\
&- \frac{\lambda_2^2(s_2^i)^2}{2C_2V^2} - \frac{\lambda_1^2S_1^2}{2C_2V^2} - \frac{\lambda_2^2S_2^2}{2C_2V^2} - \frac{\lambda_3^2x^2}{2C_2V^2} \\
&- \frac{\lambda_4^2p^2}{2C_1V^2} - \frac{\lambda_5^2q^2}{2C_1V^2} - \frac{\lambda_4^2p^2}{2C_2V^2} - \frac{\lambda_5^2q^2}{2C_2V^2} \\
&+ \frac{\lambda_1\lambda_2 s_1^i S_2}{C_1V^2} + \frac{\lambda_1^2 S_1 s_1^i}{C_1V^2} + \frac{\lambda_1\lambda_3 s_1^i x}{C_1V^2} + \frac{\lambda_1\lambda_4 s_1^i p}{C_1V^2} \\
&+ \frac{\lambda_1\lambda_5 s_1^i q}{C_1V^2} + \frac{\lambda_1\lambda_2 S_1 s_2^i}{C_2V^2} + \frac{\lambda_2^2 S_2 s_2^i}{C_2V^2} + \frac{\lambda_2\lambda_3 s_2^i x}{C_2V^2} \\
&+ \frac{\lambda_2\lambda_4 s_2^i p}{C_2V^2} + \frac{\lambda_2\lambda_5 s_2^i q}{C_2V^2} - \frac{\lambda_1\lambda_2 S_1S_2}{C_1V^2} - \frac{\lambda_1\lambda_3 S_1 x}{C_1V^2} \\
&- \frac{\lambda_2\lambda_3 S_2 x}{C_1V^2} - \frac{\lambda_1\lambda_4 S_1 p}{C_1V^2} - \frac{\lambda_2\lambda_4 S_2 p}{C_1V^2} - \frac{\lambda_3\lambda_4 xp}{C_1V^2} \\
&- \frac{\lambda_1\lambda_5 S_1 q}{C_1V^2} - \frac{\lambda_2\lambda_5 S_2 q}{C_1V^2} - \frac{\lambda_3\lambda_5 xq}{C_1V^2} - \frac{\lambda_4\lambda_5 pq}{C_1V^2} \\
&- \frac{\lambda_1\lambda_2 S_1S_2}{C_2V^2} - \frac{\lambda_1\lambda_3 S_1 x}{C_2V^2} - \frac{\lambda_2\lambda_3 S_2 x}{C_2V^2} - \frac{\lambda_1\lambda_4 S_1 p}{C_2V^2} \\
&- \frac{\lambda_2\lambda_4 S_2 p}{C_2V^2} - \frac{\lambda_3\lambda_4 xp}{C_2V^2} - \frac{\lambda_1\lambda_5 S_1 q}{C_2V^2} - \frac{\lambda_2\lambda_5 S_2 q}{C_2V^2} \\
&- \frac{\lambda_3\lambda_5 xq}{C_2V^2} - \frac{\lambda_4\lambda_5 pq}{C_2V^2}
\end{aligned}$$

4) State and costate differential equations

$$\begin{aligned}
\dot{S}_1^* &= -\rho_{max}\frac{S_1}{K_{S_1}+S_1}x - \frac{\lambda_1(s_1^i)^2}{C_1V^2} - \frac{\lambda_1S_1^2}{C_1V^2} - \frac{\lambda_1S_1^2}{C_2V^2} \\
&+ \frac{\lambda_2 s_1^i S_2}{C_1V^2} + \frac{2\lambda_1 S_1 s_1^i}{C_1V^2} + \frac{\lambda_3 s_1^i x}{C_1V^2} + \frac{\lambda_4 s_1^i p}{C_1V^2} + \frac{\lambda_5 s_1^i q}{C_1V^2} \\
&+ \frac{\lambda_2 S_1 s_2^i}{C_1V^2} - \frac{\lambda_2 S_1 S_2}{C_1V^2} - \frac{\lambda_3 S_1 x}{C_1V^2} - \frac{\lambda_4 S_1 p}{C_1V^2} - \frac{\lambda_5 S_1 q}{C_1V^2} \\
&- \frac{\lambda_2 S_1 S_2}{C_2V^2} - \frac{\lambda_3 S_1 x}{C_2V^2} - \frac{\lambda_4 S_1 p}{C_2V^2} - \frac{\lambda_5 S_1 q}{C_2V^2}
\end{aligned}$$

$$\begin{aligned}
\dot{S}_2^* &= \frac{-1}{Y_{x/s}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}x \\
&- \frac{1}{Y_{p/s}}\pi_{max}\frac{S_2}{K_{S_2}+S_2}x - k_m x - \frac{\lambda_2 S_2^2}{C_1V^2} - \frac{\lambda_2(s_2^i)^2}{C_2V^2} \\
&- \frac{\lambda_2 S_2^2}{C_2V^2} + \frac{\lambda_1 s_1^i S_2}{C_1V^2} + \frac{\lambda_1 S_1 s_2^i}{C_2V^2} + \frac{2\lambda_2 S_2 s_2^i}{C_2V^2} + \frac{\lambda_3 s_2^i x}{C_2V^2} \\
&+ \frac{\lambda_4 s_2^i p}{C_2V^2} + \frac{\lambda_5 s_2^i q}{C_2V^2} - \frac{\lambda_1 S_1 S_2}{C_1V^2} - \frac{\lambda_3 S_2 x}{C_1V^2} - \frac{\lambda_4 S_2 p}{C_1V^2} \\
&- \frac{\lambda_5 S_2 q}{C_1V^2} - \frac{\lambda_1 S_1 S_2}{C_2V^2} - \frac{\lambda_3 S_2 x}{C_2V^2} - \frac{\lambda_4 S_2 p}{C_2V^2} - \frac{\lambda_5 S_2 q}{C_2V^2}
\end{aligned}$$

$$\begin{aligned}
\dot{x}^* &= \mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}x - \frac{\lambda_3 x^2}{C_1V^2} - \frac{\lambda_3 x^2}{C_2V^2} \\
&+ \frac{\lambda_1 s_1^i x}{C_1V^2} + \frac{\lambda_2 s_2^i x}{C_2V^2} - \frac{\lambda_1 S_1 x}{C_1V^2} - \frac{\lambda_2 S_2 x}{C_1V^2} - \frac{\lambda_4 xp}{C_1V^2} \\
&- \frac{\lambda_5 xq}{C_1V^2} - \frac{\lambda_1 S_1 x}{C_2V^2} - \frac{\lambda_2 S_2 x}{C_2V^2} - \frac{\lambda_4 xp}{C_2V^2} - \frac{\lambda_5 xq}{C_2V^2}
\end{aligned}$$

$$\begin{aligned}
\dot{p}^* &= \pi_{max}\frac{S_2}{K_{S_2}+S_2}x - \frac{1}{Y_{x/p}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}x \\
&- \frac{\lambda_4 p^2}{C_1V^2} - \frac{\lambda_4 p^2}{C_2V^2} + \frac{\lambda_1 s_1^i p}{C_1V^2} + \frac{\lambda_2 s_2^i p}{C_2V^2} - \frac{\lambda_1 S_1 p}{C_1V^2} \\
&- \frac{\lambda_2 S_2 p}{C_1V^2} - \frac{\lambda_3 xp}{C_1V^2} - \frac{\lambda_5 pq}{C_1V^2} - \frac{\lambda_1 S_1 p}{C_2V^2} - \frac{\lambda_2 S_2 p}{C_2V^2} \\
&- \frac{\lambda_3 xp}{C_2V^2} - \frac{\lambda_5 pq}{C_2V^2}
\end{aligned}$$

$$\begin{aligned}
\dot{q}^* &= \rho_{max}\frac{S_1}{K_{S_1}+S_1}x - \frac{1}{Y_{x/q}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}x \\
&- \frac{\lambda_5 q^2}{C_1V^2} - \frac{\lambda_5 q^2}{C_2V^2} + \frac{\lambda_1 s_1^i q}{C_1V^2} + \frac{\lambda_2 s_2^i q}{C_2V^2} - \frac{\lambda_1 S_1 q}{C_1V^2} \\
&- \frac{\lambda_2 S_2 q}{C_1V^2} - \frac{\lambda_3 xq}{C_1V^2} - \frac{\lambda_4 pq}{C_1V^2} - \frac{\lambda_1 S_1 q}{C_2V^2} - \frac{\lambda_2 S_2 q}{C_2V^2} \\
&- \frac{\lambda_3 xq}{C_2V^2} - \frac{\lambda_4 pq}{C_2V^2}
\end{aligned}$$

$$\begin{aligned}
\dot{\lambda}_1^* &= (\lambda_1\rho_{max}x - \lambda_5\rho_{max}x)\left(\frac{1}{K_{S_1}+S_1} - \frac{S_1}{(K_{S_1}+S_1)^2}\right) + \frac{\lambda_1^2 S_1}{C_1V^2} + \frac{\lambda_1\lambda_2 S_2}{C_1V^2} + \frac{\lambda_1\lambda_3 x}{C_1V^2} \\
&+ \frac{\lambda_1\lambda_4 p}{C_1V^2} + \frac{\lambda_1\lambda_5 q}{C_1V^2} + \frac{\lambda_1^2 S_1}{C_2V^2} + \frac{\lambda_1\lambda_2 S_2}{C_2V^2} + \frac{\lambda_1\lambda_3 x}{C_2V^2} \\
&+ \frac{\lambda_1\lambda_4 p}{C_2V^2} + \frac{\lambda_1\lambda_5 q}{C_2V^2} - \frac{\lambda_1^2 s_1^i}{C_1V^2} - \frac{\lambda_1\lambda_2 s_2^i}{C_2V^2}
\end{aligned}$$

$$\dot{\lambda}_2^* = \lambda_2\left(\frac{1}{Y_{x/s}}\mu_{max}\frac{q}{K_q+q}x - \frac{1}{Y_{p/s}}\pi_{max}\right)$$

$$\left(\frac{1}{K_{S_2}+S_2}-\frac{S_2}{(K_{S_2}+S_2)^2}\right)$$
$$-\lambda_3\left(\mu_{max}\frac{q}{K_q+q}x\right)\left(\frac{1}{K_{S_2}+S_2}-\frac{S_2}{(K_{S_2}+S_2)^2}\right)$$
$$-\lambda_4\left(\pi_{max}x-\frac{1}{Y_{x/p}}\mu_{max}\frac{q}{K_q+q}x\right)\left(\frac{1}{K_{S_2}+S_2}-\frac{S_2}{(K_{S_2}+S_2)^2}\right)$$
$$+\lambda_5\left(\frac{1}{Y_{x/q}}\mu_{max}\frac{q}{K_q+q}x\right)\left(\frac{1}{K_{S_2}+S_2}-\frac{S_2}{(K_{S_2}+S_2)^2}\right)$$
$$+\frac{\lambda_2^2S_2}{C_1V^2}+\frac{\lambda_2^2S_2}{C_2V^2}-\frac{\lambda_1\lambda_2s_1^i}{C_1V^2}-\frac{\lambda_2^2s_2^i}{C_2V^2}+\frac{\lambda_1\lambda_2S_1}{C_1V^2}$$
$$+\frac{\lambda_2\lambda_3x}{C_1V^2}+\frac{\lambda_2\lambda_4p}{C_1V^2}+\frac{\lambda_2\lambda_5q}{C_1V^2}+\frac{\lambda_1\lambda_2S_1}{C_2V^2}+\frac{\lambda_2\lambda_3x}{C_2V^2}$$
$$+\frac{\lambda_2\lambda_4p}{C_2V^2}+\frac{\lambda_2\lambda_5q}{C_2V^2}$$

$$\dot{\lambda}_3^*=-1+\lambda_1\left(\rho_{max}\frac{S_1}{K_{S_1}+S_1}\right)$$
$$-\lambda_2\left(\frac{-1}{Y_{x/s}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}-\frac{1}{Y_{p/s}}\pi_{max}\frac{S_2}{K_{S_2}+S_2}-k_m\right)$$
$$-\lambda_3\left(\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}\right)$$
$$-\lambda_4\left(\pi_{max}\frac{S_2}{K_{S_2}+S_2}-\frac{1}{Y_{x/p}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}\right)$$
$$-\lambda_5\left(\rho_{max}\frac{S_1}{K_{S_1}+S_1}-\frac{1}{Y_{x/q}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}\frac{q}{K_q+q}\right)$$
$$+\frac{\lambda_3^2x}{C_1V^2}+\frac{\lambda_3^2x}{C_2V^2}-\frac{\lambda_1\lambda_3s_1^i}{C_1V^2}-\frac{\lambda_2\lambda_3s_2^i}{C_2V^2}+\frac{\lambda_1\lambda_3S_1}{C_1V^2}$$
$$+\frac{\lambda_2\lambda_3S_2}{C_1V^2}+\frac{\lambda_3\lambda_4p}{C_1V^2}+\frac{\lambda_3\lambda_5q}{C_1V^2}+\frac{\lambda_1\lambda_3S_1}{C_2V^2}+\frac{\lambda_2\lambda_3S_2}{C_2V^2}$$
$$+\frac{\lambda_3\lambda_4p}{C_2V^2}+\frac{\lambda_3\lambda_5q}{C_2V^2}$$

$$\dot{\lambda}_4^*=1+\frac{\lambda_4^2p}{C_1V^2}+\frac{\lambda_4^2p}{C_2V^2}-\frac{\lambda_1\lambda_4s_1^i}{C_1V^2}-\frac{\lambda_2\lambda_4s_2^i}{C_2V^2}$$
$$+\frac{\lambda_1\lambda_4S_1}{C_1V^2}+\frac{\lambda_2\lambda_4S_2}{C_1V^2}+\frac{\lambda_3\lambda_4x}{C_1V^2}+\frac{\lambda_4\lambda_5q}{C_1V^2}+\frac{\lambda_1\lambda_4S_1}{C_2V^2}$$
$$+\frac{\lambda_2\lambda_4S_2}{C_2V^2}+\frac{\lambda_3\lambda_4x}{C_2V^2}+\frac{\lambda_4\lambda_5q}{C_2V^2}$$

$$\dot{\lambda}_5^*=\lambda_2\left(\frac{1}{Y_{x/s}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}x\right)\left(\frac{1}{K_q+q}-\frac{q}{(K_q+q)^2}\right)$$
$$-\lambda_3\left(\mu_{max}\frac{S_2}{K_{S_2}+S_2}x\right)\left(\frac{1}{K_q+q}-\frac{q}{(K_q+q)^2}\right)$$
$$+\lambda_4\left(\frac{1}{Y_{x/p}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}x\right)\left(\frac{1}{K_q+q}-\frac{q}{(K_q+q)^2}\right)$$
$$+\lambda_5\left(\frac{1}{Y_{x/q}}\mu_{max}\frac{S_2}{K_{S_2}+S_2}x\right)\left(\frac{1}{K_q+q}-\frac{q}{(K_q+q)^2}\right)$$
$$+\frac{\lambda_5^2q}{C_1V^2}+\frac{\lambda_5^2q}{C_2V^2}-\frac{\lambda_1\lambda_5s_1^i}{C_1V^2}-\frac{\lambda_2\lambda_5s_2^i}{C_2V^2}+\frac{\lambda_1\lambda_5S_1}{C_1V^2}$$
$$+\frac{\lambda_2\lambda_5S_2}{C_1V^2}+\frac{\lambda_3\lambda_5x}{C_1V^2}+\frac{\lambda_4\lambda_5p}{C_1V^2}+\frac{\lambda_1\lambda_5S_1}{C_2V^2}+\frac{\lambda_2\lambda_5S_2}{C_2V^2}$$
$$+\frac{\lambda_3\lambda_5x}{C_2V^2}+\frac{\lambda_4\lambda_5p}{C_2V^2}$$

where the initial condition is $S_1(0)=S_{10}$, $S_2(0)=S_{20}$, $x(0)=x_0$, $p(0)=p_0$, $q(0)=q_0$; and the final condition is $\lambda_1(t_f)=0$, $\lambda_2(t_f)=0$, $\lambda_3(t_f)=0$, $\lambda_4(t_f)=0$, $\lambda_5(t_f)=0$.

However, the solution of state and costate differential equations are hard to find analytically. Thus we use DOTcvpSB to solve this numerically.

C. *Numerical Simulation*

In this section, we discuss the numerical simulations of the model. First, we perform simulation for the model without controlling the feeding strategy. Second, we perform simulation with the optimal control of the model. Then, we compare the results of those simulations. Initial conditions that are used in this numerical simulations are $S_1(0)=29.3$, $S_2(0)=0.57$, $x(0)=10$, $p(0)=2$ and $q(0)=2$. These initial conditions describe the concentration of the state variables at time $t_0=0$. Table I shows the values of parameters in the model [4].

TABLE I
PARAMETERS OF MICROALGAE GROWTH IN FED-BATCH BIOREACTOR MODEL.

Parameter	Value	Unit
$Y_{x/s}$	0.55	–
$Y_{p/s}$	0.34	–
$Y_{x/q}$	56.67	–
$Y_{x/p}$	11.84	–
k_m	0.19	L/day
μ_{max}	14.18	L/day
π_{max}	0.50	L/day
ρ_{max}	0.93	L/day
K_{S_1}	0.14	gr/L
K_{S_2}	8.45	gr/L
K_q	0.0041	gr/L
s_1^i	0.6	gr/L
s_2^i	40	gr/L
V	2.00	L

Figure 1 shows a simulation of the model without controlling the feeding strategy. In this case, the concentration of glycine, glucose, biomass, lipid, and nitrogen at the end of cultivation, are 10.08 gr/L, -0.02 gr/L, 8.55 gr/L, 1.83 gr/L, and 18.28 gr/L, respectively. Finally the value of objective function is -22.59.

TABLE II
RESULTS OF THE CONTROLLED MICROALGAE GROWTH MODEL.

Weight	State variables					
	S_1	S_2	x	p	q	J
$C_1 = 1, C_2 = 1$	0.100	0.10	11.49	0.81	17.18	-24.88
$C_1 = 1, C_2 = 2$	0.099	0.10	11.47	0.82	17.27	-24.61
$C_1 = 2, C_2 = 2$	0.100	0.10	11.49	0.81	17.18	-24.61
$C_1 = 2, C_2 = 3$	0.099	0.13	11.67	0.78	16.82	-24.20
$C_1 = 3, C_2 = 2$	0.099	0.10	11.49	0.81	17.18	-24.59

Figures 2 and 3 show a simulation of optimal control of the model. By controlling the feeding strategy, we obtain the concentration of glycine, glucose, biomass, lipid, and nitrogen at the end of cultivation, which are 0.10 gr/L, 0.10 gr/L, 11.49 gr/L, 0.81 gr/L, and 17.18 gr/L, respectively. The value of objective function is -24.88.

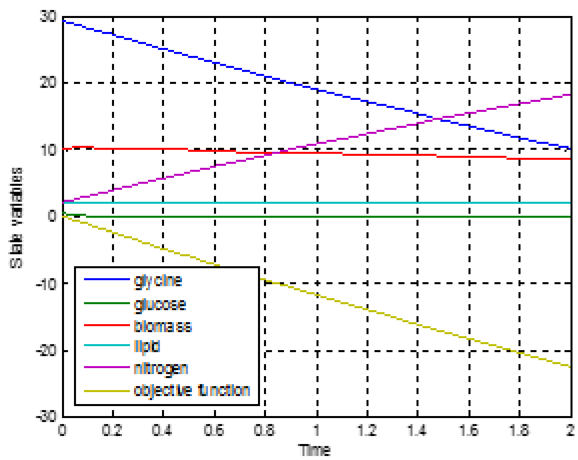

Fig. 1. Numerical solutions of the microalgae growth model without controlling the feeding strategy.

Figure 3 describes the optimal feeding strategy of the model. The feeding rate is adjusted with respect to the problem at hand. The optimal glycine feeding rate is constant 0.1 mL/day. The optimal glucose feeding rate is decreased from 0.81 mL/day, 0.71 mL/day, 0.63 mL/day, 0.56 mL/day, 0.50 mL/day, 0.43 mL/day, 0.34 mL/day, to 0.29 mL/day.

Figures 1-3 are obtained by choosing $C_1 = 1$ and $C_2 = 1$. We present the results of the controlled model for some other values of C_1 and C_2 in Table II.

V. CONCLUSIONS

The solution of optimal control problem of microalgae growth model can be formulated by Pontryagin Minimum Principle, and simulated by DOTcvpSB. However, for some values of C_1 and C_2, the numerical simulation shows the decrease of biomass concentration. It implies that the values of control weight on objective function effect the optimization result.

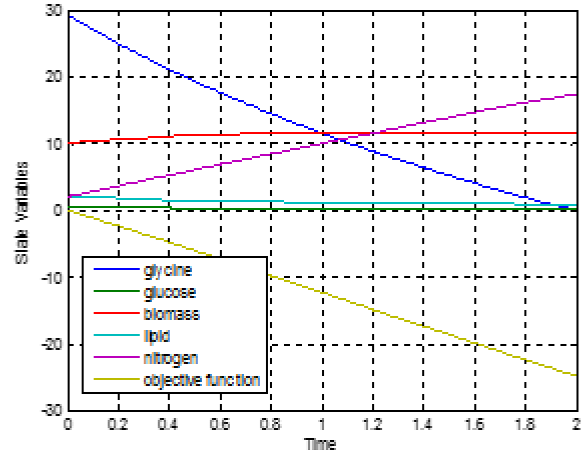

Fig. 2. Numerical solutions of the microalgae growth model with optimal feeding strategy.

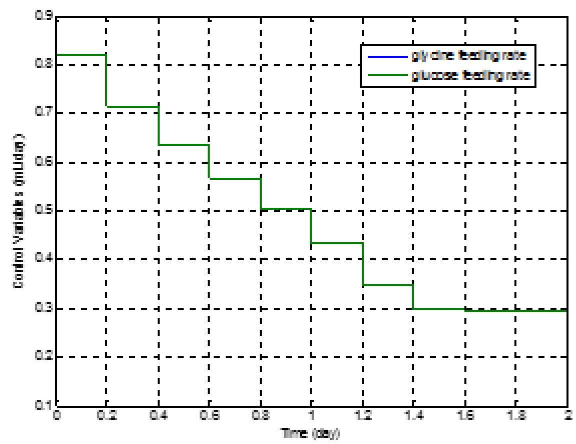

Fig. 3. Figure 3. Optimal feeding strategy of the microalgae growth model.

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