Convergence and Completeness in $\mathcal{L}_2(\mathbf{P})$ with respect to a Partial Metric

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***Abstract*—Metric spaces can be generalized to partial metric spaces. Partial metric spaces have a unique concept related to a distance. In usual case, there is no distance from two same points. But, we can obtain the distance from two same points in partial metric spaces. It means that the distance is not absolutely zero. Using the basic concept of partial metric spaces, we find analogy between metric spaces and partial metric spaces. We define a metric d^p formed by a partial metric p, with applying characteristics of metric and partial metric. At the beginning, we implement the metric d^p to determine sequences in $\mathcal{L}_2(P)$. We then ensure the convergence and completeness in $\mathcal{L}_2[a,b]$ can be established in $\mathcal{L}_2(P)$. In this study, we conclude that the convergence and completeness in $\mathcal{L}_2[a,b]$ can be established in $\mathcal{L}_2(P)$ by constructing a partial metric p_2 induced by a metric d^p.**

***Index Terms*—Completeness, Convergence, Partial Metric.**

I. Introduction

METRIC spaces can be generalized to some spaces. Partial metric space is one such form of the generalization with implementing properties of metric spaces. A concept of partial metric space was first demonstrated by Matthews in 1994 [1]. Matthews described that the distance of two same points is not absolutely zero. By constructing the characteristics of metric and partial metric, Matthews found that a partial metric can generate a metric. Matthews defined a function composed by a partial metric and proved that the function satisfying all properties as a metric. It is an analogy between metric spaces and partial metric spaces. The results of Matthews's research motivated Heckmann [2] to extend the concepts of partial metric. In a partial metric space, Heckmann constructed some sequences. Then, Heckmann add a set P presenting a partially ordered set to establish sequences in a partial metric space. Waszkiewicz [3] later continues the study about partial metric space especially regarding the role of partially ordered set. The concept of partially ordered set is also used by Han, et al [4] to construct partially metrizable spaces. In the other hand, Wu and Yue [5] explore partially ordered set of formal balls in fuzzy partial metric spaces.

In 2013, Kadak, et al [6] determine some sequences according to the concept of partial metric related to partial ordering. Their research is based on recent studies given by [1], [2] and [7]. The results of Kadak, et al's research represented that convergence in metric space needed to guarantee convergence in partial metric space. Hereafter, Esi, et al [8] investigate the properties of metric and partial metric spaces to develop convergence concept on partial metric spaces. We use a common metric d^p to ensure convergence in a metric space. We initially define a partial metric, called p, to induce a metric d^p. Therefore, we get to maintain the convergence based on partial metric space concept.

There are some complete metric spaces. One of them is space $\mathcal{L}_2[a,b]$. Sequences in the space is assured to converge [9]. In this study, we identify that convergence in space $\mathcal{L}_2[a,b]$ can guarantee convergence in space $\mathcal{L}_2(P)$ regarding to a partial metric induced by a metric d^p. Furthermore, we ensure that space $\mathcal{L}_2(P)$ is a complete partial metric space.

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Manuscript received December 2, 2022; accepted January 27, 2023.

II. Preliminaries

A. Partial Metric Space

A concept of partial metric spaces first introduced by Matthews [1]. In a partial metric, a distance from a point to itself need not be zero. Concepts of partial metric spaces are derived from concepts of metric spaces. There is a bit difference between metric and partial metric spaces. Given two points, x and y, which $x = y$. The distance $d(x,y)$ is absolutely zero in the case of metric space. This term is not necessarily hold in the case of partial metric space. A principle of partial metric can be constructed using that term. The principle states that the distance of two points, $p(x,y)$, is not absolutely zero although the two points are same, notated by $x = y$. Matthews [1] then extended metric axioms to determine partial metrics.

Definition 1. *Given a function named p on a nonempty set X. The function p defined by $p : X \times X \to \mathbb{R}^+ \cup \{0\}$ is called a partial metric if for $x, y, z \in X$,*

(P1) $x = y$ *if and only if* $p(x,x) = p(x,y) = p(y,y)$;
(P2) $p(x,x) \le p(x,y)$;
(P3) $p(x,y) = p(y,x)$; *and*
(P4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

A partial metric space (X, p) is a pair in which X is a nonempty set and p is a partial metric on X.

An example of partial metric commonly used is a function $p(x,y) = \max\{x,y\}$, for all $x, y \in \mathbb{R}^+ \cup \{0\}$ with $p : \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$. We use the common partial metric to enquire that generalization of metric can be presented into a partial metric.

B. Metric and Partial Metric

An analogy between metrics and partial metrics can be constructed. Partial metric spaces are generalization of metric spaces. In order to guarantee that some metric's characteristics are still valid in partial metric space, an analogy between metric and partial metric is needed. A partial metric $p(x, y) = \max\{x, y\}$ forming a function, called d^p, describes an analogy of metric and partial metric as presented on Definition 2.

Definition 2. *A function d^p is defined on a nonempty set X with $d^p : X \times X \to \mathbb{R}$ for all $x, y \in X$*

$$d^p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

which p is a partial metric.

A proof that the function d^p is a metric on a nonempty set X is given by [1].

Theorem 1. *A function d^p is defined on a nonempty set X with $d^p : X \times X \to \mathbb{R}$ for all $x, y \in X$,*

$$d^p(x, y) = 2p(x, y) - p(x, x) - p(y, y). \quad (1)$$

If p is a partial metric on X, then the function d^p is a metric on X.

A distance of two points can be presented as a function $d^p(x, y) = |x - y|$. Convergence and completeness in a partial metric space can be assured using an analogy of metric and partial metric.

C. Partially Ordered Set

A partially ordered set is defined by Matthews [1] considering the principles of partial metric spaces.

Definition 3. *A pair $(X, \sqsubseteq)$ is noted as partially ordered set (poset). Set X is a nonempty set and notation $\sqsubseteq$ is represented as a partial ordering on X. In partial metric spaces with each partial metric $p : X \times X \to \mathbb{R}^+ \cup \{0\}$, a binary relation $\sqsubseteq_p$ is defined over X. For all $x, y \in X$, a relation $x \sqsubseteq_p y$ is satisfied if and only if $p(x, x) = p(x, y)$.*

Based on Definition 3, Matthews [1] proves that a binary relation $\sqsubseteq_p$ is a partial ordering in the case of partial metric spaces with each partial metric p. It is shown by Theorem 2.

Theorem 2. If $\sqsubseteq_p$ is a binary relation in partial metric space (X, p), then the binary relation $\sqsubseteq_p$ is a partial ordering for each partial metric p.

Proof: It can be shown using the three properties below that binary relation $\sqsubseteq_p$ for all partial metric p forms a partial ordering:

(i) Reflexivity: $x \sqsubseteq_p y$ for $x, y \in X$;
(ii) Antisymmetry: Equation $x = y$ is satisfied for $x, y \in X$ when $x \sqsubseteq_p y$ and $y \sqsubseteq_p x$;
(iii) Transitivity: Relation $x \sqsubseteq_p z$ is satisfied for $x, y, z \in X$ when $x \sqsubseteq_p y$ and $y \sqsubseteq_p z$.

■

Matthews [1] also defines $\max\{a, b\}$ (or $\min\{a, b\}$) as a partial metric over the nonnegative reals. A partial ordering for the partial metric is notated as $\sqsubseteq_{\max}$ (or $\sqsubseteq_{\min}$). A relation $[a, b] \sqsubseteq_p [c, d]$ is satisfied in the concept of intervals if and only if $[c, d]$ is a subset of $[a, b]$.

D. Space $\mathcal{L}_2[a, b]$

Space $\mathcal{L}_2[a, b]$ is a complete metric space. In the space, for all $f, g \in \mathcal{L}_2[a, b]$, the metric $d : \mathcal{L}_2[a, b] \times L_2[a, b] \to \mathbb{R}$ defined as follow [9]:

$$d(f, g) = \left(\int_a^b |f(x) - g(x)|^2 \, dx \right)^{\frac{1}{2}}. \quad (2)$$

Space $\mathcal{L}_2[a, b]$ is the completion of metric space $C[a, b]$. From (2), we have that the metric d in space $\mathcal{L}_2[a, b]$ represents the distance of two functions in the usual case.

E. Space $(C[a, b],\ p_1)$

Kadak, et al [6] define a partial metric in the space $C[a, b]$. The partial metric is induced by metric d^p.

Proposition 1. *Let $f, g \in C[a, b]$. If the distance of two functions in space $C[a, b]$ represented as the distance function $p_1 : C[a, b] \times C[a, b] \to \mathbb{R}^+ \cup \{0\}$ defined by*

$$p_1(f, g) = \int_a^b d^p(f(x), g(x)) \, dx, \quad (3)$$

then the distance function p_1 is a partial metric in the space $C[a, b]$.

Space $(C[a, b], p_1)$ is an incomplete partial metric space. From Part D, we know that space $\mathcal{L}_2[a, b]$ is the completion of space $C[a, b]$ refers to a metric d. Furthermore, space $\mathcal{L}_2(P)$ is the completion of space $C[a, b]$ refers to a partial metric. Notice that a metric d^p induce the partial metric.

III. RESULTS AND DISCUSSION

We describe some steps and theorems to identify the convergence and completeness in $\mathcal{L}_2(P)$ regarding a partial metric. In this case, we define partially ordered set P with Lebesgue measure of P, $\mu(P) > 0$. Convergence in $\mathcal{L}_2(P)$ can be ensured using the metric d^p. The same way is also used to ensure completeness of $\mathcal{L}_2(P)$. Let p be a partial metric inducing metric d^p. We need to construct the sequences in $\mathcal{L}_2(P)$ before ensuring the convergence and completeness in $\mathcal{L}_2(P)$.

A. Space $\mathcal{L}_2(P)$

Based on the research of Kadak, et al [6], we can construct the function space $\mathcal{L}_2(P)$ using the metric d^p. From Theorem 1, we get $d^p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$. We then define the metric d^p in $\mathcal{L}_2(P)$ using the same concept for constructing the metric d^p in Theorem 1. Therefore, we obtain that metric d^p in $\mathcal{L}_2(P)$ denoted as $d^p(f, g) = 2p(f, g) - p(f, f) - p(g, g)$. After that, we construct the sequences in $\mathcal{L}_2(P)$. The sequences are defined as the sequences of F with F is a set of all function sequences. It is shown by:

$$\mathcal{L}_2(P) := \left\{ f = f_i \in F : \int_P [d^p(f_i, 0)]^2 \, dx < \infty \right\}. \quad (4)$$

From (4), we can see how the form of sequences in $\mathcal{L}_2(P)$. It is a reference to construct a partial metric in $\mathcal{L}_2(P)$. The partial metric can be useful for investigating the convergence and completeness in $\mathcal{L}_2(P)$.

B. Partial Metric in $\mathcal{L}_2(P)$

We define a function in $\mathcal{L}_2(P)$. A metric d^p induces the function. In Theorem 3, we prove that a partial metric constructed in $\mathcal{L}_2(P)$ can be yield by the function. Moreover, metric in $\mathcal{L}_2(P)$ can be presented as the function

Theorem 3. *Let P be a partially ordered set. If for all $f, g \in \mathcal{L}_2(P)$, the function $p_2 : \mathcal{L}_2(P) \times \mathcal{L}_2(P) \to \mathbb{R}^+ \cup \{0\}$ defined as*

$$p_2(f,g) = \left(\int_P [d^p(f,g)]^2 \, dx \right)^{\frac{1}{2}}. \tag{5}$$

then the function p_2 is a partial metric as well as a metric in $\mathcal{L}_2(P)$.

Proof: We prove the theorem using the partial metric axioms shown by Definition 1 and metric axioms presented by Teschl [9]. At the beginning of this paper, we know that partial metrics are generalization of metrics. Consequently, some of partial metric axioms are generalization of metric axioms. It is shown by the axiom (P1) which is generalization of the axiom (D2). Moreover, the axiom (P2) is generalization of the axiom (D1), whereas the axiom (P4) is generalization of the axiom (D4). Furthermore, there is an axiom of partial metric which has the same description with an axiom of metric. The partial metric axiom is (P3) and the metric axiom is (D3).

Therefore, we prove the theorem as follow:

- It is shown that the function p_2 satisfies the axiom (P1) and (D2).
  $(\Rightarrow)$
  If $f = g$, then $p_2(f,f) = p_2(f,g) = p_2(g,g) = 0$.
  $(\Leftarrow)$
  Since Lebesgue measure of P, $\mu(P) > 0$, therefore if $p_2(f,f) = p_2(f,g) = p_2(g,g)$, then $f = g$.
- It is described that the function p_2 satisfies the axiom (P2) and (D1). In the other word, it is proved that $p_2(f,f) = 0 \leq p_2(f,g)$.
- It is explained that the function p_2 satisfies the axiom (P3) and (D3).
- It is proved that the function p_2 satisfies the axiom (P4) and (D4). It is enough to show that $p_2(f,h) \leq p_2(f,g) + p_2(g,h)$.

The function p_2 defined by (5) satisfies partial metric axioms and metric axioms. It means that the function is a partial metric as well as a metric in $\mathcal{L}_2(P)$. ■

Theorem 3 shows that the function p_2 on (5) states a partial metric induced by metric d^p. Metric d^p represents a usual metric. If the distance function in the space $\mathcal{L}_2[a,b]$ is described as the quadratic integral from the distance of two functions in usual case (the difference of two functions). Then, the distance function in the space $\mathcal{L}_2(P)$ is represented as the quadratic integral of metric d^p. In this paper, we choose a metric d^p in a function space $C[a,b]$. In the space $C[a,b]$, the distance function is defined as a metric d with $d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|$. The metric d is a usual metric described the distance between two functions in general. Therefore, the metric d can be denoted as the metric d^p.

From Theorem 1, a partial metric p can induce a metric d^p. In the proposition below, we define a partial metric p related to a partially ordered set P. Any closed interval set can be determined as partially ordered set in the space $\mathcal{L}_2(P)$.

Proposition 2. *Let $P \subseteq \mathbb{R}$ is any closed interval set. If the function $p : P \times P \to \mathbb{R}^+ \cup \{0\}$ for all $[s,t],[u,v] \in P$, is defined by*

$$p([s,t],[u,v]) = \max\{|s-u|, |t-v|\} \tag{6}$$

then the function p denotes a partial metric on $P \subseteq \mathbb{R}$.

Proof: We use partial metric axioms given by Definition 1 for proving the proposition

(P1) It is shown that
$(\Rightarrow)$
If $[s,t] = [u,v]$, then
$p([s,t],[s,t]) = p([s,t],[u,v]) = p([u,v],[u,v])$.
$(\Leftarrow)$
If $p([s,t],[s,t]) = p([s,t],[u,v]) = p([u,v],[u,v])$, then $[s,t] = [u,v]$.

(P2) It is explained that $p([s,t],[s,t]) \leq p([s,t],[u,v])$.

(P3) It is identified that $p([s,t],[u,v]) = p([u,v],[s,t])$.

(P4) It is proved that
$p([s,t],[m,n]) \leq p([s,t],[u,v]) + p([u,v],[m,n]) - p([u,v],[u,v])$.
We consider some conditions for proving the inequality. The conditions is presented as follow:

- For $[s,t]$ and $[m,n]$, we obtain four criteria:
  a) $[s,t] \cap [m,n] = \varnothing$
  b) $[s,t] \cap [m,n] \neq \varnothing$, with $[s,t] \nsubseteq [m,n]$ and $[m,n] \nsubseteq [s,t]$
  c) $[s,t] \subseteq [m,n]$
  d) $[m,n] \subseteq [s,t]$
- For $[s,t]$ and $[u,v]$, we have four criteria:
  a) $[s,t] \cap [u,v] = \varnothing$
  b) $[s,t] \cap [u,v] \neq \varnothing$, with $[s,t] \nsubseteq [u,v]$ and $[u,v] \nsubseteq [s,t]$
  c) $[s,t] \subseteq [u,v]$
  d) $[u,v] \subseteq [s,t]$
- For $[u,v]$ and $[m,n]$, we get four criteria:
  a) $[u,v] \cap [m,n] = \varnothing$
  b) $[u,v] \cap [m,n] \neq \varnothing$, with $[u,v] \nsubseteq [m,n]$ and $[m,n] \nsubseteq [u,v]$
  c) $[u,v] \subseteq [m,n]$
  d) $[m,n] \subseteq [u,v]$

We let $x = [s,t], y = [u,v]$, and $z = [m,n]$ for doing an investigation in a closed interval set $P \subseteq \mathbb{R}$.

The result is the function p on (6) represents a partial metric. ■

In Proposition 1, we have that a metric on P can be expressed as the function p given by (6). It is shown by Moore, et al [10]. Consequently, the partial metric on Proposition 2 can be represented as the usual metric in space $C[a,b]$. It presents that there is a partial metric as well as a metric. On Proposition 3, we prove that the usual metric in space $C[a,b]$ is a partial metric.

Proposition 3. *In a partially ordered set P, a partial metric $p : P \times P \to \mathbb{R}^+ \cup \{0\}$, for all $[s,t],[u,v] \in P$, defined as*

$$p\left([s,t],[u,v]\right) = \max\left\{|s-u|,|t-v|\right\} \tag{7}$$

is represented as a usual metric in space $C[a,b]$ with $d : C[a,b] \times C[a,b] \to \mathbb{R}$, for all $f,g \in C[a,b]$,

$$d\left(f,g\right) = \max_{x\in[a,b]}\{|f(x)-g(x)|\}, \tag{8}$$

such that the metric d is a partial metric in space $C[a,b]$.

Proof: From Proposition 2, we have proved that the function p defined by (7) is a partial metric. Then, we show that the partial metric can be represented as a usual metric in space $C[a,b]$. First, we get two functions from the set of real numbers $\mathbb{R}$ denoted as $f(x)$ and $g(x)$. For all $x \in [a,b]$, the function $f(x)$ defined by $f : [a,b] \to \mathbb{R}$ and the function $g(x)$ defined by $g : [a,b] \to \mathbb{R}$. On (7), we have that $s,t,u,v \in \mathbb{R}$ present values. Therefore, they can represent the values of the function f and g. Let $s = f(a), t = f(b), u = g(a)$ and $v = g(b)$, such that (7) becomes

$$\begin{aligned} &p\left(\left[f\left(a\right),f\left(b\right)\right],\left[g\left(a\right),g\left(b\right)\right]\right)\\ &=\max\left\{\left|f\left(a\right)-\ g\left(a\right)\right|,\left|f\left(b\right)-\ g\left(b\right)\right|\right\}.\end{aligned} \tag{9}$$

We can rewrite (9) as

$$\begin{aligned} &p\left(\left[f\left(a\right),f\left(b\right)\right],\left[g\left(a\right),g\left(b\right)\right]\right)\\ &=\max_{x\in[a,b]}\left\{\left|f\left(x\right)-\ g\left(x\right)\right|\right\}.\end{aligned} \tag{10}$$

From (10), we can identify that the function p given by (7) is a usual metric in space $C[a,b]$ described on (8). After that, we investigate that the function p defined by (8) satisfies the partial metric axioms. ■

In Proposition 4, we prove that the closed interval set P with the partial ordering $\sqsubseteq_p$ is a partially ordered set $P = (P, \sqsubseteq_p)$ which p given by (7).

Proposition 4. *A closed interval set P with the partial ordering v_p denoted as $(P, \sqsubseteq_p)$, is a partially ordered set for a partial metric $p : P \times P \to \mathbb{R}^+ \cup \{0\}$, which for all $[s,t],[u,v] \in P$ defined by*

$$p\left([s,t],[u,v]\right) = \max\left\{|s\ -\ u|,|t\ -\ v|\right\}. \tag{11}$$

Proof: We first show that p, the function on (11), is a partial metric. On Proposition 2, we have proved that the function denotes a partial metric. Next, we identify that the closed interval set given by Proposition 4 is a partially ordered set. We identify it using the three properties on the proof of Theorem 2. ■

Proposition 4 describes that for any partial metric, both the partial metric which is not a metric and the partial metric as well as a metric, satisfies the partial ordering $\sqsubseteq_p$.

In this part, we have explained about the partial metric in space $\mathcal{L}_2(P)$ in accordance with a partially ordered set. Then, we examine the convergence in space $\mathcal{L}_2(P)$ using the partial metric on (5).

C. Convergence in Space $\mathcal{L}_2(P)$

We first consider the convergence in space $\mathcal{L}_2[a,b]$. A sequence $\{f_n\}$ in space $\mathcal{L}_2[a,b]$ is said to converge if for all $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, satisfies

$$d^p\left(f_n,f\right) = \left(\int_P [|f_n(x)-f(x)|]^2\,dx\right)^{\frac{1}{2}} < \varepsilon \tag{12}$$

On (12), we denote $f_n = \{f_{n_i}\}$ and $f = \{f_i\}$ for all $i \in \mathbb{N}$. From the equation, the metric d^p describes the distance of two functions in usual case.

Next, we show that for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $p_2(f_n,f) < \varepsilon$ for all $n > n_0$ with $f \in \mathcal{L}_2(P)$ and P is a partially ordered set. From (5), we get that a partial metric p_2 is a quadratic integral of the metric d^p. In this paper, we choose the metric d^p induced by a partial metric p on (11). The metric d^p in this study has the same representation with the metric d^p in space $\mathcal{L}_2[a,b]$. Therefore, we obtain that

$$p_2(f_n,f) = \left(\int_P [d^p(f_n,f)]^2\,dx\right)^{\frac{1}{2}} < \varepsilon. \tag{13}$$

Equation (13) explains that the sequence $\{f_n\}$ converges in space $\mathcal{L}_2(P)$. It is a consequence of the convergence in space $\mathcal{L}_2[a,b]$. This is presented on Corollary 1.

Corollary 1. *A sequence $\{f_n\}$ in space $\mathcal{L}_2(P)$ converges to $f \in \mathcal{L}_2(P)$ if for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $p_2(f_n,f) < \varepsilon$ for all $n > n_0$.*

After ensuring the convergence in space $\mathcal{L}_2(P)$, we then identify the completeness of space $\mathcal{L}_2(P)$.

D. Completeness of Space $\mathcal{L}_2(P)$

We consider the completeness of space $\mathcal{L}_2[a,b]$ to investigate the completeness of $\mathcal{L}_2(P)$. Space $\mathcal{L}_2[a,b]$ is a complete metric space. It means that every Cauchy sequence converges in the space. Let for each $i \in \mathbb{N}$, $\{f_n\} = \{f_{n_i}\}$ be any Cauchy sequence in space $\mathcal{L}_2[a,b]$. Then, we obtain that for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $d^p(fn,fm) < \varepsilon$ for all $n,m > n_0$. As consequence of the existence of sequence $\{f_n\}$ in space $\mathcal{L}_2[a,b]$, we get that the sequence converges to $f = \{f_i\} \in \mathcal{L}_2[a,b]$. In the other words, the sequence satisfies (12).

We then ensure that space $\mathcal{L}_2(P)$ is a complete space with respect to a partial metric p_2. Let $\{f_n\}$ be any Cauchy sequence in space $\mathcal{L}_2(P)$. Then, we obtain that for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n,m > n_0$,

$$p_2\left(f_n,f_m\right) = \left(\int_P \left[d^p\left(f_{n_i},f_{m_i}\right)\right]^2\ dx\right)^{\frac{1}{2}} < \varepsilon. \tag{14}$$

On (14), we get for each fixed $i \in \mathbb{N}$ that for all $n,m > n_0$,

$$d^p\left(f_{n_i},f_{m_i}\right) <\ \varepsilon. \tag{15}$$

From (15), we know that the sequence $\{f_n\}$ is also in space $\mathcal{L}_2[a,b]$. By considering (14), we have for all $n,m > n_0$ that

$$\int_P \left[d^p\left(f_{n_i},f_{m_i}\right)\right]^2\ dx < \varepsilon^2. \tag{16}$$

If we take for any $n > n_0$ from (16), then we obtain for $m \to \infty$ that

$$\int_P [d^p(f_{n_i}, f_i)]^2 \, dx < \varepsilon^2. \tag{17}$$

We rewrite (17) as

$$\left(\int_P [d^p(f_{n_i}, f_i)]^2 \, dx \right)^{\frac{1}{2}} < \varepsilon. \tag{18}$$

Equation (18) describes that for every $\varepsilon > 0$, there is $n \in n_0$, such that for $n > n_0$ yields

$$p_2(f_n, f) = \left(\int_P [d^p(f_{n_i}, f_i)]^2 \, dx \right)^{\frac{1}{2}} < \varepsilon. \tag{19}$$

From (19), we get that the Cauchy sequence $\{f_n\}$ converges to f. After that, we identify that $f \in \mathcal{L}_2(P)$.

We realize that the metric d^p on (18) has the same representation with the metric d^p given by (12). Consequently, we obtain that $f_n - f = \{f_{n_i} - f_i\} \in \mathcal{L}_2(P)$. By using Minkowski inequality, we have

$$\begin{aligned} &\left(\int_P [d^p(f_i, 0)]^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_P [d^p(f_i, f_{n_i})]^2 \, dx \right)^{\frac{1}{2}} + \left(\int_P [d^p(f_{n_i}, 0)]^2 \, dx \right)^{\frac{1}{2}} \\ &< \infty. \end{aligned} \tag{20}$$

Equation (20) explains that $f = \{f_i\} \in \mathcal{L}_2(P)$. From the investigation we have done, we obtain a corollary about the completeness of space $\mathcal{L}_2(P)$.

Corollary 2. *Space $\mathcal{L}_2(P)$ is a complete space with repect to a partial metric p_2 defined as*

$$p_2(f, g) = \left(\int_P [d^p(f, g)]^2 \, dx \right)^{\frac{1}{2}}$$

for all $f, g \in \mathcal{L}_2[a, b]$.

IV. CONCLUSION

We can construct sequences in space $\mathcal{L}_2(P)$ using a metric d^p. We also have presented that sequences converge in space $\mathcal{L}_2(P)$ with respect to a partial metric p_2 induced by a metric d^p. By considering the completeness of space $\mathcal{L}_2[a, b]$, we obtain that space $\mathcal{L}_2(P)$ is a complete partial metric space. It concludes that the convergence and completeness in space $\mathcal{L}_2[a, b]$ can be established in space $\mathcal{L}_2(P)$.

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