Further Results on P_h-supermagic Trees

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***Abstract*—Let G be a simple, finite, and undirected graph. An H-supermagic labeling is a bijective map $f : V(G) \cup E(G) \rightarrow \{1, 2, \cdots, |V(G)| + |E(G)|\}$ in which $f(V) = \{1, 2, \cdots, |V(G)|\}$ and there exists an integer m such that $w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = m$, for every subgraph $H' \cong H$ in G. In this paper, we determine some classes of trees which have P_h-supermagic labeling.**

***Index Terms*—Magic labeling, subgraph covering, trees.**

I. Introduction

LET G be a simple, finite, and undirected graph. Let two isomorphic graphs G and G' are denoted by $G \cong G'$. A graph G is called a tree if it does not contain any cycles. An amalgamation of a collection of graphs $\{G_i\}$ is obtained by picking a vertex to be a terminal in each of G_i, and identifying the graphs by their terminals [1]. We use the notation $Amal\{(G_i), c\}$ for an amalgamation which is obtained from identifying all c from each G_i. For convenience, let $[a, b] = \{i \mid i \in \mathbb{N}, a \leq i \leq b\}$.

Let H be a subgraph of G. If the graph G has a property that each edges of G belongs to at least one subgraph isomorphic to H in G, we say G admits H-covering. A bijection $f : V(G) \cup E(G) \rightarrow [1, |V(G) + E(G)|]$ is called H-magic labeling if there exists an integer m such that $w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = m$, for every subgraph $H' \cong H$ of G. If for every vertex $v \in V$, $f(v) \in [1, |V(G)|]$, then f may also be called H-supermagic. The problem is to determine whether a certain graph admits H-magic or H-supermagic labeling.

Some known results are found for the subgraph H is isomorphic to either a star $K_{1,n}$, a cycle C_n, and a path P_n. Gutiérrez and Lladó [2] have found that $K_{1,n}$ is $K_{1,h}$-supermagic if $h \in [1, n]$ and $K_{n,n}$ is $K_{1,n}$-magic. Roswitha and Baskoro [3] have determined some double stars, caterpillars, fire crackers, and banana trees to be $K_{1,h}$-magic for some h. Moreover, some known graphs which are C_h-magic (or supermagic) for some h are found which include wheels, windmills [4], fans, books, ladders [5], and jahangirs [6].

Furthermore, some results of P_h-magic (or supermagic) graphs are also determined. Gutiérrez and Lladó [2] found that paths are P_h-supermagic as follows.

Theorem 1: Let $n \geq 3$ be an integer. The path P_n is P_h-supermagic for any $h \in [2, n]$.

Next, let $h \geq 3$ be an integer. We define *grass* graph to be the class graph of trees that admits P_h-covering such that all subgraphs P_h in the graph contain identical vertex, denoted by $Rb(h)$. A *center* of grass graph is the identical vertex of every subgraph P_h. We may write an equivalent theorem by [7] as follows.

Theorem 2: Let $h \geq 2$ be an integer. Any graph belongs to $Rb(h)$ is P_h-supermagic.

The definition of $Rb(h)$ may be used to find a radius of a tree graph. A radius of a tree graph $r(G)$ may be defined as $r(G) = m - 1$ where m is the least number h such that the graph admits P_h-covering.

Known results are discussing not only sufficient conditions of P_h-magic graphs but also necessary conditions of P_h-magic graphs. One of the results is determined by Maryati et al. [8] stating that P_h-magic graphs cannot contain a subgraph H_n constructed as follows. Let $n \geq 1$ be an integer. Obtain two disjoint odd paths P_{2n+1} and add one more edge such that the center of those two graphs are adjacent.

Theorem 3: Let n be a positive integer and $h \in [1, n]$. If G is a P_{h+2}-magic graph, then G is H_n-free.

Some other P_h-magic (or supermagic) graphs are shackles and amalgamations [1], disjoint union of graphs and amalgamations [9], and cycles with some pendants [8]. Variants for this problem can be seen in [10], [11] and for more information of H-magic (or supermagic) labeling, please consult to [12]. In this paper, we would like to investigate more about P_h-supermagic tree graphs.

II. Main Results

Denote e_v^c to be an edge which belongs to a path from v to c and incident to v. We start this section by introducing an useful lemma.

Lemma 1: Let $h \geq 2$ be an integer, and G be a P_h-magic tree with a magic labeling f where $t = |V(G)|$. If there exists a subgraph H which belongs to $Rb(h)$ with c as a center, such that every pair $v_i \in H$ and its incident edge $e_{v_i}^c$ satisfy $f(v_i) + f(e_{v_i}^c) = 2t + 1$ then for arbitrary H' which belongs to $Rb(h)$ with a center c', $G' \cong Amal\{(G, H'), c'\}$ is P_h-magic. Also, if G is P_h-supermagic, then G' is P_h-supermagic.

Proof: Denote $n = |V(H')| - 1$, or equivalent of total vertices in H' without its center. Let f' be a labeling of G'. Then, for every $v \in V(G)$ and $e \in E(G)$, label as follows

$$f'(v) = f(v), \qquad f'(e) = f(e) + 2n.$$

Take all the unused labels $\{t+1, t+2, ..., t+2n\}$ and create a partition into 2-sets, sets consists of two elements, such that the sum of the elements of each 2-set is $2t + 2n + 1$. Then, use all these 2-sets to label all $\{v_i, e_{v_i}^c\}$ in any order so that

$$f'(v_i) + f'(e_{v_i}^c) = 2t + 2n + 1.$$

with $f'(v_i) < f'(e_{v_i}^c)$.

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By evaluating, for every subgraph P_h we got $w(P_h) = (h-1)(2t+2n+1) + f'(c)$. ∎

This lemma enable us to identify the center of any $Rb(h)$ to a terminal vertex of P_h-magic graph in order to produce other P_h-magic (or supermagic) graph. The terminal vertex chosen for this study is mostly a pendant.

Theorem 4: Let $h \geq 3$ be an integer and let H belongs to $Rb(h)$. The graph $G \cong Amal\{(H, P_{h+1}, P_{h+1}), c\}$ with c is a center of H and a pendant of each P_{h+1} is P_h-supermagic.

Proof: Denote $V(G) = V(H) \cup \{u_i, v_i \mid i \in [1, h+1]\}$ with $u_1 = v_1 = c$ and $E(G) = E(H) \cup \{u_iu_{i+1}, v_iv_{i+1} \mid i \in [1, h]\}$. Let $t = |V(G)|$. Define a labeling f as follows

$$\begin{aligned} f(u_2) &= 5, & f(cu_2) &= 2t-4, \\ f(u_{h+1}) &= 1, & f(u_hu_{h+1}) &= 2t-1, \\ f(v_2) &= 6, & f(cv_2) &= 2t-5, \\ f(v_{h+1}) &= 2, & f(v_hv_{h+1}) &= 2t-3, \\ f(c) &= 4. \end{aligned}$$

Compile the unused labels $\{3\} \cup \{7, ..., t\} \cup \{t+1, ..., 2t-6\} \cup \{2t-2\}$ and create a partition of 2-sets such that the sum of the elements of each 2-sets is $2t+1$. Then, use all these 2-sets to label all unlabeled pairs $\{v_i, e^c_{v_i}\}, \{u_i, e^c_{u_i}\}$ and $\{q, e^c_q\}$ for $q \in H$ in any order such that

$$f(v_i) + f(e^c_{v_i}) = f(u_i) + f(e^c_{u_i}) = f(q) + f(e^c_q) = 2t+1.$$

with $f(v_i) < f(e^c_{v_i}), f(u_i) < f(e^c_{u_i})$, and $f(q) < f(e^c_q)$. By Lemma 1, G is P_h-supermagic. ∎

An example of a tree for Theorem 4 can be seen in Figure 1.

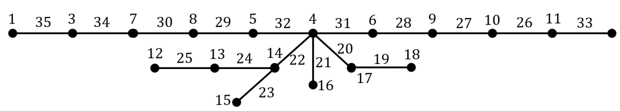

Fig. 1. A P_5-supermagic tree.

Theorem 5: Let $h \geq 3$ be an integer and let H belongs to $Rb(h)$. The graph $G \cong Amal\{(H, P_{h+1}, P_{h+1}, P_{h+1}), c\}$ with c is a center of H and a pendant of each P_{h+1} is P_h-supermagic.

Proof: Denote $V(G) = V(H) \cup \{u_i, v_i, x_i \mid i \in [1, h+1]\}$ with $u_1 = v_1 = x_1 = c$ and $E(G) = E(H) \cup \{u_iu_{i+1}, v_iv_{i+1}, x_ix_{i+1} \mid i \in [1, h]\}$. Let $t = |V(G)|$. Define a labeling f as follows

$$\begin{aligned} f(u_2) &= 2, & f(cu_2) &= 2t-1, \\ f(u_{h+1}) &= 7, & f(u_hu_{h+1}) &= 2t-6, \\ f(v_2) &= 4, & f(cv_2) &= 2t-3, \\ f(v_{h+1}) &= 6, & f(v_hv_{h+1}) &= 2t-5, \\ f(x_2) &= 5, & f(cx_2) &= 2t-4, \\ f(x_{h+1}) &= 1, & f(x_hx_{h+1}) &= 2t-2, \\ f(c) &= 3. \end{aligned}$$

Compile the unused labels and create a partition of 2-sets such that the sum of the elements of each 2-sets is $2t+1$. Then, use all these 2-sets to label all unlabeled pairs $\{u_i, e^c_{u_i}\}, \{v_i, e^c_{v_i}\}, \{x_i, e^c_{x_i}\}$ and $\{q, e^c_q\}$ for $q \in H$ in any order such that

$$\begin{aligned} & f(v_i) + f(e^c_{v_i}) = f(u_i) + f(e^c_{u_i}) \\ =& f(x_i) + f(e^c_{x_i}) = f(q) + f(e^c_q) = 2t+1. \end{aligned}$$

with $f(v_i) < f(e^c_{v_i}), f(u_i) < f(e^c_{u_i}), f(x_i) < f(e^c_{x_i})$ and $f(q) < f(e^c_q)$. By Lemma 1, G is P_h-supermagic. ∎

Theorem 6: Let $h \geq 3$ be an integer and let H belongs to $Rb(h)$. The graph $G \cong Amal\{(H, P_{h+1}, P_{h+2}), c\}$ with c is a center of H and a pendant both of P_{h+1} and P_{h+2} is P_h-supermagic.

Proof: Denote $V(G) = V(H) \cup \{u_i, v_i \mid i \in [1, h+1]\} \cup \{v_{h+2}\}$ with $u_1 = v_1 = c$ and $E(G) = E(H) \cup \{u_iu_{i+1}, v_iv_{i+1} \mid i \in [1, h]\} \cup \{v_{h+1}v_{h+2}\}$. Let $t = |V(G)|$. Define a labeling f as follows

$$\begin{aligned} f(u_2) &= 6, & f(cu_2) &= 2t-5, \\ f(u_{h+1}) &= 1, & f(u_hu_{h+1}) &= 2t-2, \\ f(v_2) &= 8, & f(cv_2) &= 2t-7, \\ f(v_3) &= 5, & f(v_2v_3) &= 2t-4, \\ f(v_{h+1}) &= 3, & f(v_hv_{h+1}) &= 2t-6, \\ f(v_{h+2}) &= 7, & f(v_{h+1}v_{h+2}) &= 2t-3, \\ f(c) &= 4. \end{aligned}$$

Compile the unused labels and create a partition of 2-sets such that the sum of the elements of each 2-sets is $2t+1$. Then, use all these 2-sets to label all unlabeled pairs $\{v_i, e^c_{v_i}\}, \{u_i, e^c_{u_i}\}$ and $\{q, e^c_q\}$ for $q \in H$ in any order such that

$$f(v_i) + f(e^c_{v_i}) = f(u_i) + f(e^c_{u_i}) = f(q) + f(e^c_q) = 2t+1.$$

with $f(v_i) < f(e^c_{v_i})$, $f(u_i) < f(e^c_{u_i})$, and $f(q) < f(e^c_q)$. By Lemma 1, G is P_h-supermagic. ∎

Before we continue to the next theorem, we need define to define P^+_n. Let y be one of a vertices in a path P_h which is adjacent to a pendant. For $n \geq 4$, a graph P^+_n is obtained from a path P_n which the vertex y is attached with one more pendant. A pendant z of P^+_n is called a *furthest* pendant if it is not adjacent to y.

Theorem 7: Let $h \geq 3$ be an integer and let H of order at least two belongs to $Rb(h)$. The graph $G \cong Amal\{(H, P^+_n, P_n), c\}$ with c is a center of H, and a (furthest) pendant of both P_h and P^+_h is P_h-supermagic.

Proof: Since the order of H is at least two, there exists a subgraph K_2 in H such that $V(H) = V(H') \cup \{x\}$ and $E(H) = E(H') \cup \{cx\}$ for some other H' which belongs to $Rb(h)$. Denote $V(G) = V(H) \cup \{u_i, v_i \mid i \in [1, h+1]\} \cup \{u^+_h\}$ with $u_1 = v_1 = c$ and $E(G) = E(H) \cup \{u_iu_{i+1}, v_iv_{i+1} \mid i \in [1, h]\} \cup \{u_hu^+_h\}$. Let $t = |V(G)|$ and $r = |V(H)| - 2$. Define a labeling f as follows

$$\begin{aligned} f(u_i) &= h - i + 2, & & i \in [1, h], \\ f(u_iu_{i+1}) &= 2t - h + i, & & i \in [1, h-1], \\ f(u_{h+1}) &= h+2, & f(u_hu_{h+1}) &= 2t - h, \\ f(u^+_h) &= h+3, & f(u_hu^+_h) &= 2t - h - 1, \\ f(v_2) &= t - r, & f(cv_2) &= t + r + 1, \end{aligned}$$

$$f(v_{h+1}) = 1, \qquad f(v_h v_{h+1}) = t + h + r + 1.$$

Compile the unused labels and create a partition of 2-sets such that the sum of the elements of each 2-sets is $2t+1$. Then, use all these 2-sets to label all unlabeled pairs $\{v_i, e^c_{v_i}\}$, $\{u_i, e^c_{u_i}\}$, and $\{q, e^c_q\}$ for $q \in H$ in any order such that

$$f(v_i) + f(e^c_{v_i}) = f(u_i) + f(e^c_{u_i}) = f(q) + f(e^c_q) = 2t + 1$$

with $f(v_i) < f(e^c_{v_i})$, $f(u_i) < f(e^c_{u_i})$, and $f(q) < f(e^c_q)$. By Lemma 1, G is P_h-supermagic. ■

An example of a tree in Theorem 7 is illustrated in Figure 2.

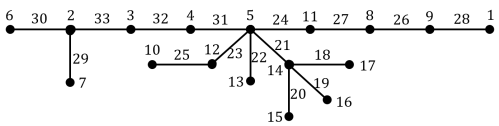

Fig. 2. A P_4-supermagic tree.

Theorem 8: Let $h \geq 2$ be an integer and let H, H' belongs to $Rb(h)$. If $G' \cong Amal\{(H, P_h), c\}$, where c is a center of H and a pendant of P_h, then the amalgamation $G \cong Amal\{(G', H'), c'\}$ where c' is a center of H' and the other pendant of P_h is P_h-supermagic.

Proof: First, we need to prove $G' \cong Amal\{(H, P_h), c\}$ is P_h-supermagic with a magic labeling satisfying the condition of the lemma. Denote $V(G') = V(H) \cup \{u_i | i \in [1, h]\}$ where $c = u_1$, $p = u_h$ and $E(G') = E(H) \cup \{u_i u_{i+1} | i \in [1, h-1]\}$. Let $t = |V(G')|$. Label the vertices and edges as follows

$$f(u_i) = t - i + 1, \qquad f(u_i u_{i+1}) = t + i.$$

Then, take $\{1, 2, 3, ..., t-h\} \cup \{t+h, t+2, ..., 2t-1\}$, and create a partition of 2-sets such that the sum of the elements of each 2-set is $2t$. Use all these 2-sets to label all $\{v_i, e^c_{v_i}\}$ in any order so that

$$f(v_i) + f(e^c_{v_i}) = 2t$$

where $f(v_i) < f(e^c_{v_i})$. By evaluating, for every P_h we got $w(P_h) = (h-1)(2t) + f(c)$.

We have shown that $G' \cong Amal\{(H, P_h), c\}$ is P_h-supermagic with a magic labeling f. It can be seen that there exists a subgraph H^* of G' which belongs to $Rb(h)$ with u_h as a center. This subgraph and the magic labeling f are satisfying Lemma 1, hence by applying the lemma, we have $Amal\{(G', H'), c'\}$ with $c' = u_h$ is P_h-supermagic. ■

Furthermore, the next result is applicable for $h = 3$.

Theorem 9: Let H_1, H_2 be graphs which belongs to $Rb(3)$. Then, $Amal\{(H_1, H_2), p\}$ where p is a pendant of both H and H' is P_3-supermagic.

Proof: Let $|V(H_k)| = t_k$ and c_k are the centers of H_k for $k \in \{1, 2\}$. The proof is divided into two cases based on $d(c_i, p)$.

Case 1: $d(c_1, p) = d(c_2, p)$

Let

$$A = [1, t_1 - 2] \cup [t_1 + 2t_2, 2(t_1 + t_2) - 3],$$

$$B = [t_1 + 2, t_1 + 2t_2 - 3]$$

Create a partition for A and B into 2-sets such that the sum of the elements of each 2-set is $2(t_1 + t_2 - 1)$ for A and $2(t_1 + t_2) - 1$ for B. Construct a f labeling as follows

$$f(p) = t_1,$$

$$f(c_1) = t_1 + 1,$$

$$f(c_2) = t_1 - 1,$$

$$f(e^{c_1}_p) = t_1 + 2t_2 - 2,$$

$$f(e^{c_2}_p) = t_1 + 2t_2 - 1.$$

Use all 2-sets from A to label all $\{v, e^{c_1}_v\}$ for $v \in H_1$ in any order so that

$$f(v) + f(e^{c_1}_v) = 2(t_1 + t_2 - 1)$$

with $f(v) < f(e^{c_1}_v)$. Again, use all 2-sets from B to label all $\{u, e^{c_2}_u\}$ for $u \in H_2$ in any order so that

$$f(u) + f(e^{c_2}_u) = 2(t_1 + t_2) - 1$$

where $f(u) < f(e^{c_2}_u)$. Therefore, every vertices have smaller labels from every edges. Furthermore, for every P_3 we got $f(P_3) = 4(t_1 + t_2) + 6$.

Case 2: $d(c_1, p) \neq d(c_2, p)$

Without loss of generality, $d(c_1, p) < d(c_2, p)$. Let

$$A = [1, t_1 - 3] \cup [t_1 + 2t_2 + 1, 2(t_1 + t_2) - 3],$$

$$B = [t_1 - 1, t_1 + t_2 - 4] \cup [t_1 + t_2 + 1, t_1 + 2t_2 - 2]$$

Create a partition for each A and B into 2-sets such that the sum of the elements of each 2-set is $2(t_1 + t_2 - 1)$ for A and $2(n_1 + n_2) - 3$ for B. Construct a f labeling as follows

$$f(p) = t_1 - 2,$$

$$f(c_1) = t_1 + t_2 - 3,$$

$$f(c_2) = t_1 + t_2 - 1,$$

$$f(e^{c_1}_p) = t_1 + 2t_2,$$

$$f(e^{c_2}_p) = t_1 + 2t_2 - 1$$

Choose $v_1 \in H_1$ other than c_1 or p. Continue labels as follows

$$f(v_1) = n_1 + n_2 - 2,$$

$$f(e^{c_1}_{v_1}) = t_1 + t_2.$$

Use all 2-sets from A to label all $\{v, e^{c_1}_v\}$ for $v \in H_1, v \neq v_1$ in any order so that

$$f(v) + f(e^{c_1}_v) = 2(t_1 + t_2 - 1)$$

with $f(v) < f(e^{c_1}_v)$. Again, use all 2-sets from B to label all $\{u, e^{c_2}_u\}$ for $u \in H_2$ in any order so that

$$f(u) + f(e^{c_2}_u) = 2(t_1 + t_2) - 3$$

with $f(u) < f(e^{c_2}_u)$. Therefore, every vertices have smaller labels from every edges. Furthermore, for every P_3 we got $f(P_3) = 5(t_1 + t_2) - 7$.

Hence, the theorem holds. ■

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