In Search of Dotless Kropki Puzzle Solution

Andri Purnama Ramadan

***Abstract*—Searching all possible solution and finding the minimum number of clues to make uniquely solvable puzzle always been a natural question for puzzle enthusiast. However, the attempt usually provide that as difficult task. In this paper, we attempt to search the solution of Kropki puzzle without dot clues given with graph theory approach, which resulted in some conjectures involving the planarity of graph and cyclicity of latin square.**

***Index Terms*—Kropki, Latin Square, Puzzle**

I. Introduction

Latin square is an arrangement of n uniquely different elements into $n \times n$ array such that no row and column contains repeating element. Even though latin square can use arbitrary symbol for the element, latin square puzzles like sudoku, kropki, and futoshiki usually use the element of $A = \{1, 2, \dots, n\}$ for their element symbols.

Studying the solution or number of clues of puzzle is not a new thing. Some works of it like [1]–[4] had been published, even proof of minimum clues such that sudoku can be uniquely solvable had been given by [5] with 7.1 million hours of computation claim. However, research on latin square puzzle seems mostly still involved in sudoku. It's not surprising because of sudoku's popularity, but it's encourage us to study other latin square puzzle.

Kropki is a latin square puzzle with additional constraint using black and white dots between two cells. Every adjacent cells with $1:2$ number ratio should be indicated with black dots and every adjacent cells with consecutive number should be indicated with white dots. Thus, if between the adjacent cells contain no black or white dots, then those numbers must not be consecutive or have $1:2$ ratio.

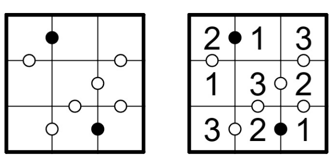

Fig. 1: Kropki puzzle and its solution

Like the name, dotless kropki means there are no dot clue given, so every adjacent cells should not contain any consecutive number nor the numbers have $1:2$ ratio.

II. Preliminaries

In this paper, we write $n \times n$ latin square as n-ordered latin square. Every rows and columns' label will begin from 0 consecutively from top to bottom and left to right. For example, if we have 3-ordered latin square, then we will have row 0, row 1, and row 2 from top to bottom and column 0, column 1, and column 2 from left to right, labelling the said row and column.

Every binary operation involving row and column will be operated in $\mathbb{Z}_n$ even if it's not implicitly stated. However, notice that even though we use the element of $A = \{1, 2, \dots, n\}$ to fill the Kropki puzzle, we will use them only as a symbol without making any operation of it, so it shouldn't make any trouble.

Definition 1. *Let L be n-ordered latin square. The element of row r and column c from L is notated as $x_{r,c}$.*

To shorten the writing, we will write row r, column c, and element $x_{r,c}$ as 3-tuple $(r, c, x_{r,c})$, which then we will call it coordinate of L.

Definition 2. *[6] Let L be n-ordered latin square. If there exist $k \in \mathbb{N}$ such that $x_{r,c} = x_{r+1,c+k}$ for every r and c, then L is called k-cyclic.*

Considering the global constraint of dotless kropki puzzle about what number can be adjacent to other number, we can also construct the relation of the numbers using graph.

Definition 3. *n-ordered dotless kropki graph is $G = (V, E)$ with $V = \{1, 2, \dots, n\}$ and $E = \{(v_i, v_j) : v_i, v_j$ nonconsecutive and the ratio of them is not $1:2\}$ for distinct $i, j \in \{0, 1, \dots, n-1\}$*

Let $v_0, v_1, \dots, v_k$ a sequence of distinct vertices in $G = (V, E)$ and $P_1 = (V_1, E_1)$ where $V_1 = \{v_0, v_1, \dots, v_k\} \subseteq V(G)$ and $E_1 = \{(v_i, v_{i+1}) : i \in 0, 1, \dots, k-1\} \subseteq E(G)$. We call P as path in G. If edge (v_0, v_k) exists in G, then $C = (V_1, E_1 \cup \{(v_0, v_k)\})$ is called cycle in G. Hamiltonian path is a path such that every vertices on G is visited exactly once. Similarly, Hamiltonian cycle is a Hamiltonian path with additional edge connecting path's end vertices [7].

Classifying all graph that has Hamiltonian path or cycle is not an easy task. In fact, research on finding globally sufficient condition of Hamiltonian path and cycle is still being developed like by [8], [9]. For this paper, Fan's result from [10] will be used to assist our work. We will write degree of x, which is the number of edges connected to x, as $\deg(x)$.

Definition 4. *[11] Let $G = (V, E)$ be a graph, $|G|$ be the number of vertices in G, and $k \in \mathbb{N}$. If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E$ of fewer than k edges, then G is called k-edge-connected.*

In simple word, a graph G is called k-edge-connected if G remain connected whenever we remove l amount of edges in G, with $l < k$. Furthermore, for our convenience, we will call k-edge-connected as k-connected.

A.P. Ramadan is independent researcher from Kabupaten Bandung, West Java, Indonesia. Email: (Andri.Purnama98@gmail.com).
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Theorem 1. *[10] Let $G = (V, E)$ be 2-connected graph with $n \geq 3$ vertices and let u, v be distinct vertices with distance of 2. If $max\{\deg(u), \deg(v)\} \geq \frac{n}{2}$ then G have Hamiltonian cycle.*

If there's a way to draw a graph G on Euclidian plane such that every edges are not intersecting, then G is called planar graph. One famous theorem to examine the planarity of a graph is Kuratowski's theorem.

Theorem 2 (Kuratowski's theorem)**.** *[12] Let $G = (V, E)$ be a graph. G is nonplanar if and only if it contains subdivision of complete graph K_5 or complete bipartite graph $K_{3,3}$*

We can illustrate subdividing graph as adding additional vertex on an edge, so the vertex divide the edge. As in contains, we can only examine the subgraph of nonplanar graph G and find the subdivision of K_5 or $K_{3,3}$.

Now we are going to search dotless kropki puzzle solution.

III. Results and Discussions

For $n = 1$, it's obvious that the puzzle will only have one solution. For nontrivial cases, let L be n-ordered latin square and to be filled with element of $A = \{1, 2, \ldots, n\}$. Because every rows and columns must contain non-repeating n symbols, then there exist $r, c \in \mathbb{N}$ with $c \neq 1, n$ such that $(r, c, x_{r,c})$ is coordinate of L. Thus, every $x_{r,c}$ must be adjacent to at least two different numbers. In graph, we could interpreted it as degree of $v = x_{r,c}$ should be at least 2.

For $n = 2, 3, 4$, we have $\deg(2) = 0$ because it can't be connected to $1, 3$, and 4. For $n = 5$, we have $\deg(2) = 1$ because it can only be connected to 5. Since every vertices' degree of dotless kropki puzzle should be at least 2 to have a solution, then we can conclude that n-ordered dotless kropki puzzle don't have any solution for $n = 2, 3, 4, 5$.

For $n = 6$, we will have $\deg(v) \geq 2$ for every v. If we add the next vertex consecutively, we will still have $\deg(v) \geq 2$, since the kropki constraint will only prohibit it connected to maximum 2 other previous vertices, so there are still $(n - 2)$ other vertices to be choosen and it will implies $\deg(v) \geq 2$ for every v for all $n \geq 6$.

Lemma 1. *For $n \geq 6$, every n-ordered dotless kropki graph have Hamiltonian cycle.*

Proof. Let $G = (V, E)$ be n-ordered dotless kropki graph. Since for every $v \in V(G)$ we have $\deg(v) \geq 2$, so G must be 2-connected.

We will always have $1 \in V(G)$ with $\deg(1) = (n - 1)$ for every $n \geq 6$, since 1 will be connected to every other number except 2. However, there's $5 \in V(G)$ such that $(1, 5), (2, 5) \in E(G)$ for every n, thus there will always be vertices 1 and 2 with distance 2.

Since $max\{\deg(1), \deg(2)\} = n - 1 \geq \frac{n}{2}$, then by theorem 1, G must be have Hamiltonian cycle. □

In n-ordered latin square puzzle, the solution in k-cyclic form can be constructed if n, k are coprime, and it's very clear that 1 and $n-1$ will always be coprime with n. However, since we have additional kropki constraint, we can't directly stated that easily because the relationship of their adjacency may forbid us to construct such solution. Fortunately, they are not.

Theorem 3. *Let $n \geq 6$ and L be n-ordered dotless kropki puzzle. Dotless kropki graph always have solution in 1-cyclic and $(n - 1)$-cyclic form.*

Proof. By lemma 1, we know that every dotless kropki graph have Hamiltonian cycle.

Let $v_0, v_1, \ldots, v_{n-1}, v_0$ be vertices sequence that establish the Hamiltonian cycle, then we can construct puzzle solution for row r by $(r, i, x_{r,i})$ coordinates with $x_{r,i} = v_i$. Representing it in graph, it's clear that $x_{r,i}$ must be connected to $x_{r,i-1}$ and $x_{r,i+1}$, which have been fulfilled by the Hamiltonian cycle.

Notice that for every $r, c \in \{0, 1, \ldots, (n - 1)\}$, $x_{r,c}$ is adjacent to $x_{r,c-1}, x_{r,c+1}, x_{r-1,c}$, and $x_{r+1,c}$. If the puzzle solution is in form of 1-cyclic like we already defined on definition 2, we will have $x_{r,i-1} = x_{r+1,i}$ and $x_{r,i+1} = x_{r-1,i}$. So, if we consider the row $r - 1$ and row $r + 1$, we need no any additional edges for the graph. Thus, we will not breaking any kropki rules and the solution still satisfy the kropki puzzle. Proof for $(n - 1)$ is analogous. □

By theorem 3 we conclude that n-ordered dotless kropki puzzle always have solution in 1-cyclic and $(n - 1)$-cyclic form, but are they the only solution form?

Theorem 4. *Let L be 6-ordered dotless kropki puzzle, then L only have 1-cyclic and 5-cyclic form solution.*

Proof. We already know that 6-ordered dotless kropki puzzle have solution in 1-cyclic or 5-cyclic form. So, we only need to proof there are no other form solution.

Since L is latin square, then every column should contain 2. So, we will have $(r, 1, 2)$ as solution coordinate for some r. However, notice that 2 can only be connected to $5, 6$, this implies $(r, 0, 6)$ and $(r, 2, 5)$ or $(r, 0, 5)$ and $(r, 2, 6)$ must be the other solution coordinates.

Let $(r, 0, 6)$ and $(r, 2, 5)$ be the other solution coordinates. Notice that 3 can only be connected to $1, 5$. Thus, we will have $(r, 3, 3), (r, 4, 1)$, and $(r, 5, 4)$ as another solution coordinates. So, if we look at the row r, we will have $6, 2, 5, 3, 1, 4$ sequence as solution of the puzzle on row r. On the other hand, if we let $(r, 0, 5)$ and $(r, 2, 6)$ as solution coordinates, then we will have $5, 2, 6, 4, 1, 3$ sequence as solution of the puzzle on row r.

So, the puzzle always should contain $6, 2, 5, 3, 1, 4$ or $5, 2, 6, 4, 1, 3$ sequence in some row to be solvable.

Let L have row of $6, 2, 5, 3, 1, 4$ for the solution. Considering the solution coordinates of row r, we can only have $(r+1, 0, 2)$ or $(r+1, 2, 2)$ as other solution coordinate. If we have $(r+1, 2, 2)$ as solution coordinate, we are going to have $(r+1, 0, 4), (r+1, 1, 6), (r+1, 3, 5), (r+1, 4, 3), (r+1, 5, 1)$ as other solution coordinate. This also implies $(r-1, 0, 2), (r-1, 1, 5), (r-1, 2, 3), (r-1, 3, 1), (r-1, 4, 4), (r-1, 5, 6)$ as other coordinate. Repeating this, we will have $k = 1$ such that $x_{r,c} = x_{r+1,c+1}$ for all r, c, which means the solution is 1-cyclic.

With similar approach and letting $(r+1, 0, 2)$, we will have $5 - cyclic$ solution. We will also have similar result by letting

$(r, 0, 5)$ and $(r, 2, 6)$ as solution coordinates. So, no matter how we choose the possibility, it will be always 1-cyclic or 5-cyclic. □

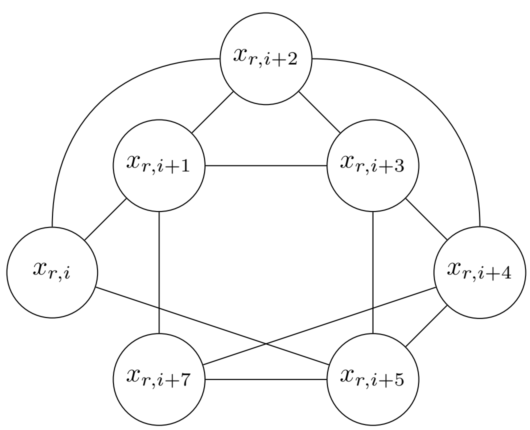
Considering the permutation of latin square row, we can easily see that there are only $2 \times 6 \times 2 = 24$ possible solution for 6-ordered dotless kropki puzzle. Unfortunately, the more vertices the dotless kropki graph have, the more complex it will be.

Theorem 5. *Let L be 7-ordered dotless kropki puzzle. If the puzzle solution is in $k-cyclic$ form, then $k = 1$ or 6*

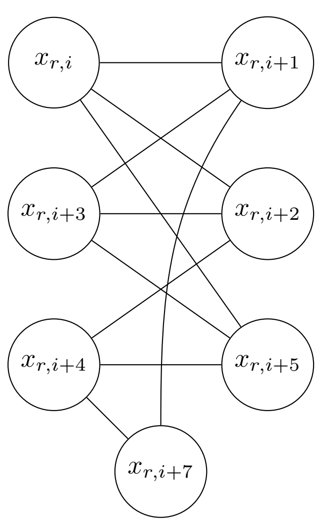
Proof. By theorem 3, we know that 7-ordered dotless kropki puzzle must have a solution in 1-cyclic and 6-cyclic form. So, Assume the solution is in $k - cyclic$ form with $k \neq 1, 6$.

As already mentioned in the proof of theorem 3, $x_{r,i}$ must be connected to $x_{r,i-1}, x_{r,i+1}, x_{r-1,i}, x_{r+1,i}$. However, since $k \neq 1, 6$, then the vertices connected to $x_{r,i}$ must be distinct.

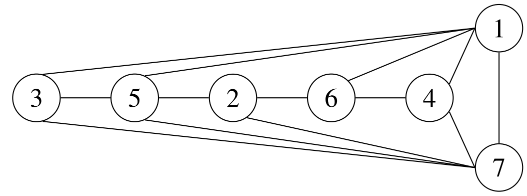
Now, let $k = 2$, then the dotless kropki graph must contain this below graph as subgraph.



Call it graph $G = (V, E)$. Now take $H = (V, E - \{(x_{r,i+1}, x_{r,i+2}), (x_{r,i+3}, x_{r,i+4}), (x_{r,i+5}, x_{r,i+6})\})$ subgraph of G, then we will have H as below



As we can see, graph H contain subdivision of $K_{3,3}$, so by theorem 2 H must be nonplanar. Since H is not planar, then G must be nonplanar. However, 7-ordered dotless kropki graph is planar as shown below.



So, it's impossible for the dotless kropki graph to contains G, therefore k must not be 2. For $k = 3, 4, 5$, similar proof will follow and it force k to be 1 or 6 only. □

An alternative proof is to show that for solution to be k-cyclic with $k \neq 1, (n-1)$, then it should satisfy $\deg(v) \geq 4$ for every vertices v, but in 7-ordered dotless kropki graph we have 2 with $\deg(2) = 3$. This is also our background to propose conjecture 1 later.

Of course the solution of dotless kropki puzzle doesn't always have to be k-cyclic. For example, we have valid non k-cyclic solution for 8-ordered dotless kropki as below.

7	4	1	3	5	8	6	2
4	1	3	7	2	6	8	5
6	8	5	2	7	4	1	3
1	5	2	6	4	7	3	8
8	3	7	4	6	2	5	1
3	7	4	1	8	5	2	6
5	2	6	8	1	3	7	4
2	6	8	5	3	1	4	7

However, Finding all possibility of solution combination is exhausting, even for $n = 7$. The cyclicity ensure degree of every vertices v of G to be $\deg(v) = 4$. Without it, it's possible to have $\deg(v) = 2, 3, 4$. If we ignore what vertices does the edge connecting to, we still have $3^7 = 2187$ possible combination to be examine. The number of combination for $n = 7$ is still low, but it will grow exponentially as n increasing.

After tedious brute force, we can't find any another solution of 7-ordered dotless kropki in any form other than 1-cyclic and 6-cyclic. We know brute force method is not practical for large number of n. So, after proving theorem 5, we try to find any properties that related to the solution classification.

8-ordered dotless kropki graph $G = (V, E)$ is non-planar because there's subgraph $H = (V - \{4, 6\}, E - \{(1, 3), (5, 7), (5, 8)\})$ which contains subdivision of $K_{3,3}$. As 8-ordered dotless kropki graph will always be subgraph of n-ordered dotless kropki graph with $n > 8$, then n-ordered dotless kropki graph must be nonplanar for $n \geq 8$.

We suspect distance of vertices pair, degree of vertices, and the planarity of graph is related to construct the dotless kropki puzzle solution, so we propose some conjecture which seems to be true.

Conjecture 1. *Let $n \geq 6$ and $G = (V, E)$ be n-ordered dotless kropki graph, then there exists $v_i \in V$ such that $\deg(v_i) < 4$ if and only if every k-cyclic solution form of the kropki puzzle only satisfied with $k = 1, (n - 1)$*

For left to right proof, it can be seen by looking through any coordinate solution $(r, c, x_{r,c})$. If we want k-cyclic solution with $k \neq 1, (n - 1)$, then $x_{r,c}$ must be adjacent with distinct

$x_{r+1,c}, x_{r-1,c}, x_{r,c+1}$, and $x_{r,c-1}$, which means $\deg(x_{r,c})$ should be at least 4. Thus, it's impossible to construct such solution with $\deg(v_i) < 4$ for some v_i. By theorem 3, the only solution in k-cyclic form are 1-cyclic and $(n-1)$-cyclic.

For right to left proof, we already prove through theorem 4 and 5 that k-cyclic solution form with $k \neq 1, (n-1)$ doesn't exists for $n = 6, 7$. And, for $n = 6, 7$, we have $\deg(2) = 3 < 4$ which hold true for the statement. To investigate further, we also search all possible combination to create k-cyclic form for $n = 8$.

Let $n = 8$. We need to find all Hamiltonian cycle from 8-ordered dotless kropki graph and use the Hamiltonian path of the cycle as our solution of row 0's puzzle. After that, we can construct our k-cyclic by shifting every elements of the row $(r-1)$ to fill row r by k and prove that there will be at least one adjacent cells which break the kropki puzzle rules.

For illustration, let's take 135274681 as our Hamiltonian cycle from 8-ordered dotless kropki graph, then we can fill the row 0 of our solution candidate with 13527468 consucetively.

1	3	5	2	7	4	6	8

Fig. 2: Our solution candidate for 8-ordered dotless kropki

Then, shifting the previous row's element by k cells with $k = 1, 2, \dots, 7$ respectively, we will have below k-cyclic form as in figure 3 from figure 2. The shaded cells are the elements that break the rules of latin square or kropki.

k = 1

1	3	5	2	7	4	6	8
8	1	3	5	2	7	4	6
6	8	1	3	5	2	7	4
4	6	8	1	3	5	2	7
7	4	6	8	1	3	5	2
2	7	4	6	8	1	3	5
5	2	7	4	6	8	1	3
3	5	2	7	4	6	8	1

k = 2

1	3	5	2	7	4	6	8
6	8	1	3	5	2	7	4
7	4	6	8	1	3	5	2
5	2	7	4	6	8	1	3
1	3	5	2	7	4	6	8
6	8	1	3	5	2	7	4
7	4	6	8	1	3	5	2
5	2	7	4	6	8	1	3

k = 3

1	3	5	2	7	4	6	8
4	6	8	1	3	5	2	7
5	2	7	4	6	8	1	3
8	1	3	5	2	7	4	6
7	4	6	8	1	3	5	2
3	5	2	7	4	6	8	1
6	8	1	3	5	2	7	4
2	7	4	6	8	1	3	5

k = 4

1	3	5	2	7	4	6	8
7	4	6	8	1	3	5	2
1	3	5	2	7	4	6	8
7	4	6	8	1	3	5	2
1	3	5	2	7	4	6	8
7	4	6	8	1	3	5	2
1	3	5	2	7	4	6	8
7	4	6	8	1	3	5	2

k = 5

1	3	5	2	7	4	6	8
2	7	4	6	8	1	3	5
6	8	1	3	5	2	7	4
3	5	2	7	4	6	8	1
7	4	6	8	1	3	5	2
8	1	3	5	2	7	4	6
5	2	7	4	6	8	1	3
4	6	8	1	3	5	2	7

k = 6

1	3	5	2	7	4	6	8
5	2	7	4	6	8	1	3
7	4	6	8	1	3	5	2
6	8	1	3	5	2	7	4
1	3	5	2	7	4	6	8
5	2	7	4	6	8	1	3
7	4	6	8	1	3	5	2
6	8	1	3	5	2	7	4

k = 7

1	3	5	2	7	4	6	8
3	5	2	7	4	6	8	1
5	2	7	4	6	8	1	3
2	7	4	6	8	1	3	5
7	4	6	8	1	3	5	2
4	6	8	1	3	5	2	7
6	8	1	3	5	2	7	4
8	1	3	5	2	7	4	6

Fig. 3: Every $k-cyclic$ form from figure 2

By theorem 3, we know that 1-cyclic and 7-cyclic form solution exist. However, from the figure 3, we can see that there are no any other solution for $k = 2, 3, \dots, 6$, at least with our choosen Hamiltonian cycle before, because it will either break the latin square rules (if k, n is coprime) or kropki rules.

Since if k, n coprime, then the k-cyclic form will not satisfy latin square rules, thus we can neglect every k that coprime with n to ease our work. So that for $n = 8$, we only need to inspect all the 3-cyclic, 5-cyclic form for our solution candidate.

The complete list of Hamiltonian path and every k-cyclic candidate solution that need to be inspected for $n = 8$ are given in our supplementary files, and all of the candidate solution will have at least one adjacent cell that broke kropki rules. Moreover, for $n = 8$ we have $\deg(4) = 3 < 4$, which again analogous with our statement.

Interestingly, for $n > 8$, it seems we will always find k-cyclic form solution with $k \neq 1, (n-1)$, which also stronger our notion for conjecture 1. Figure 4 show one of such solution for $n = 9, 10, 11, 12$.

n = 9 k = 2

1	3	5	8	2	6	9	4	7
4	7	1	3	5	8	2	6	9
6	9	4	7	1	3	5	8	2
8	2	6	9	4	7	1	3	5
3	5	8	2	6	9	4	7	1
7	1	3	5	8	2	6	9	4
9	4	7	1	3	5	8	2	6
2	6	9	4	7	1	3	5	8
5	8	2	6	9	4	7	1	3

n = 10 k = 3

1	3	5	8	10	2	6	4	7	9
4	7	9	1	3	5	8	10	2	6
10	2	6	4	7	9	1	3	5	8
3	5	8	10	2	6	4	7	9	1
7	9	1	3	5	8	10	2	6	4
2	6	4	7	9	1	3	5	8	10
5	8	10	2	6	4	7	9	1	3
9	1	3	5	8	10	2	6	4	7
6	4	7	9	1	3	5	8	10	2
8	10	2	6	4	7	9	1	3	5

n = 11 k = 2

1	3	5	7	2	9	11	4	6	10	8
10	8	1	3	5	7	2	9	11	4	6
4	6	10	8	1	3	5	7	2	9	11
9	11	4	6	10	8	1	3	5	7	2
7	2	9	11	4	6	10	8	1	3	5
3	5	7	2	9	11	4	6	10	8	1
8	1	3	5	7	2	9	11	4	6	10
6	10	8	1	3	5	7	2	9	11	4
11	4	6	10	8	1	3	5	7	2	9
2	9	11	4	6	10	8	1	3	5	7
5	7	2	9	11	4	6	10	8	1	3

n = 12 k = 5

1	3	5	2	6	4	7	9	11	8	12	10
9	11	8	12	10	1	3	5	2	6	4	7
5	2	6	4	7	9	11	8	12	10	1	3
8	12	10	1	3	5	2	6	4	7	9	11
6	4	7	9	11	8	12	10	1	3	5	2
10	1	3	5	2	6	4	7	9	11	8	12
7	9	11	8	12	10	1	3	5	2	6	4
3	5	2	6	4	7	9	11	8	12	10	1
11	8	12	10	1	3	5	2	6	4	7	9
2	6	4	7	9	11	8	12	10	1	3	5
12	10	1	3	5	2	6	4	7	9	11	8
1	7	9	11	8	12	10	1	3	5	2	6

Fig. 4: k-cyclic solution with $k \neq 1, (n-1)$ for $n = 9, 10, 11, 12$

Remember that for $n = 9, 10, 11, 12$, every degree of the vertices always equals or more than 4, and Figure 4 show that there exists k-cyclic form solution of kropki with $k \neq 1, (n-1)$, which is in line with our propose conjecture.

Furthermore, we propose our next conjecture.

Conjecture 2. *For $n \geq 6$, the solution of n-ordered dotless kropki is only in 1-cyclic and $(n-1)$-cyclic form if and only if n-ordered dotless kropki graph is planar.*

We know that dotless kropki graph is planar only if $n = 6, 7$. Through theorem 4, 5, and some small describing about finding solution through brute force, we have right to left proof.

For left to right proof, we may examine it through contra-positive statement. So, if we have nonplanar graph, then there must be other solution other than in 1-cyclic and $(n-1)$-cyclic form.

The easiest attempt to validate that it is probably by finding any $k-$cyclic solution with $k \neq 1, n-1$. However, we should find general condition so that the Hamiltonian cycle on the dotless kropki graph will preserve the condition of latin square and kropki rules to not be broken. (Thus, we automatically eliminate all possible non-coprime n, k pair, since it'll not hold latin square properties anymore). So, if we can prove conjecture 1, then conjecture 2 will be proven automatically.

Since we already found non k-cyclic solution for 8-ordered dotless kropki puzzle, we also suspect there are non k-cyclic solution for arbitary n-ordered dotless kropki puzzle, with $n \geq 8$.

For more generalization, we also propose third conjecture as below.

Conjecture 3. *Let L be latin square puzzle with some restriction of the element adjacency. If the constructed graph $G = (V, E)$ of the puzzle's elements is planar and contain Hamiltonian cycle, then the puzzle solution must be in 1-cyclic or $(n-1)$-cyclic form.*

For partial work, let's assume that L only have k-cyclic form solution with $k \neq 1, (n-1)$, let k, n coprime, and construct the solution by one row of $v_0, v_1, v_2, \ldots, v_{n-1}$ Hamiltonian path from the Hamiltonian cycle with $v_i \in G(V)$, then v_i must be connected to v_{i+1}, v_{i-1} (by Hamiltonian path) and v_{i+k}, v_{i-k} (by additional edge from $(r-1), r$, and $(r+1)$ adjacency).

Since we have n, k coprime, then we must have $v_{i+k} \neq v_{i-k}$. It's still possible that $(v_{i+k}, v_{i-k}) \in E(G)$. However, let's assume that $(v_{i+k}, v_{i-k}) \notin E(G)$ for now.

Let's take vertices $v_i, v_{i+1}, v_{i+k}, v_{i+k+1}, v_{i-k}, v_{i-k+1}$. It's obvious that v_i is connected to v_{i+1}, v_{i+k} and v_{i-k}.

Since the solution is k-cyclic, we will also have v_{i+k+1} connected to v_{i+k} (from the Hamiltonian path) and v_{i+1} (from the additional edge). With similar reasoning, we also will have v_{i-k+1} connected to v_{i+1} and v_{i-k}.

Remember that the row solution constructed from Hamiltonian cycle, thus for $v_{i+k+1} \in V(G)$, we must have path from v_{i+k+1} to v_{i-k} by $v_{i+k+1}, v_{i+k+2}, \ldots, v_{i-2k}, v_{i-k}$ through Hamiltonian path and additional edge for the last edge. Within similar reasoning, we also will have $v_{i-k+1}, v_{i-k+2}, \ldots, v_{i-1}v_{i+k-1}, v_{i+k}$ path from v_{i-k+1} to v_{i+k} without intersecting the path of v_{i+k+1} to v_{i-k}. This prove that the graph will have subdivision of $K_{3,3}$, so by theorem 2 it must be nonplanar and contradict the planarity hypothesis.

However, the same argument will not work if $(v_{i+k}, v_{i-k}) \in E(G)$, because then we will have $v_{i+k+1} = v_{i-k}$. Also, of course the works only covered some possibility since we assume that the solution is on k-cyclic. However, we hope this partial work will give insight for further study.

IV. Conclusions

We create theorem 3, 4, and 5 to find the solution of dotless kropki puzzle and propose conjecture 1, 2, and 3 as open questions from our search of dotless kropki puzzle solution.

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