Connectivity of The Triple Idempotent Graph of Ring $\mathbb{Z}_n$

Vika Yugi Kurniawan, Bayu Purboutomo, Sutrima, Nughthoh Arfawi Kurdhi

***Abstract*—Let R be a commutative ring and $I(R)$ denotes a set of all idempotent elements of R. The triple idempotent graph of ring R, denoted by $TI(R)$, is the undirected simple graph with vertex-set in $R-\{0,1\}$. Two distinct vertices u and v in $TI(\mathbb{Z}_n)$ are adjacent if and only if there exists $w \in R-\{0,1\}$ where $w \neq u$ and $w \neq v$ such as $uv \notin I(R)$, $uw \notin I(R)$, $vw \notin I(R)$ and $uvw \in I(R)$. In this research, we study the connectivity of the triple idempotent graph of ring integer modulo n, denoted by $TI(\mathbb{Z}_n)$. The result is that the triple idempotent graph of ring $\mathbb{Z}_n$ is a connected graph if n prime and $n \geq 7$.**

***Index Terms*—the triple idempotent graph, ring of integers modulo n, connected graph**

I. Introduction

LET R be a commutative ring with unit element $1 \neq 0$. In 1988, Beck [4] introduced the concept of a zero-divisor graph that connect between ring theory and graph theory. In [2], Anderson and Livingston modified zero-divisor graph, denotes by $\Gamma(R)$, with vertices $Z(R)^* = Z(R) - \{0\}$ and two distinct vertices $x, y \in Z(R)^*$ adjacent if and only if $xy = 0$. There was shown that $\Gamma(R)$ is a connected graph with $\mathrm{gr}(\Gamma(R)) \in \{3,4,\infty\}$. In other paper, Akhtar and Lee [3], studied the connectivity of the zero divisor graph $\Gamma(R)$ associated to a finite commutative ring R. They investigated the conditions of ring R such that graph $\Gamma(R)$ is a connected graph. Later, many papers that investigated various kind of graphs associated with the ring, see [1], [10], [8], and [11].

Recently in [9], Mohammad and Shuker introduced graph that is called idempotent divisor graph, denoted by $\mathrm{J}I(R)$, with the set of vertices $R^* = R - \{0\}$ and two distinct vertices v_1 and v_2 adjacent if and only if $v_1.v_2 = e$, for some non-unit idempotent element $e \in R(i.e\ e^2 = e \neq 1)$. Let $I(R)$ be a set of idempotent elements of ring R. In this paper, the definition of the triple idempotent graph of a commutative ring R, denoted by $TI(R)$, is the undirected simple graph with vertex-set $R-\{0,1\}$. Two distinct vertices u and v are in $TI(R)$ adjacent if and only if there exist $w \in R-\{0,1\}$ where $w \neq u$ and $w \neq v$ such as $uv \notin I(R)$, $uw \notin I(R)$, $vw \notin I(R)$ and $uvw \in I(R)$. We will investigate the properties that related to connectivity of the triple idempotent graph of ring integer modulo n, denoted by $TI(\mathbb{Z}_n)$.

V. Y. Kurniawan, B. Purboutomo, Sutrima, and N. A. Kurdhi are with the Universitas Sebelas Maret, Surakarta, 57126 Indonesia e-mail: vikayugi@staff.uns.ac.id

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II. Preliminaries

According to Chartrand and Zhang [7], a graph G is a finite nonempty set V of objects is called vertices together with a possibly empty set E of 2-element subsets of V is called edges. The number of vertices in a graph G is the order of G and the number of edges is the size of G. A graph of size 0 is called an empty graph. Two distinct vertices u and v said to be adjacent if there is an edge between u and v. The degree of a vertex u in a graph G is the number of vertices in G that are adjacent to u. If a path between two vertices of graph G can be found, then the graph G is connected.

If R is a ring, $Z(R)$ denotes the set of zero-divisors of R and $I(R)$ denotes the set of idempotent elements of R.

Definition 1. *Graph triple idempotent of commutative ring R, denoted by $TI(R)$, is the undirected simple graph with vertex-set $R-\{0,1\}$, and two different vertices u and v are in $TI(R)$ adjacent if and only if there exists $w \in R-\{0,1\}$ where $w \neq u$ and $w \neq v$ such that $u.v \notin I(R)$, $u.w \notin I(R)$, $v.w \notin I(R)$ and $u.v.w \in I(R)$, where $I(R)$ is a set of all idempotent elements of R.*

In the following, given an example of $TI(\mathbb{Z}_9)$.

Example 2. *Let $\mathbb{Z}_9$, with $\mathbb{Z}_9 = \{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7},\bar{8}\}$ and $I(\mathbb{Z}_9) = \{\bar{0},\bar{1}\}$. By the Definition 1, the set of vertex $V(TI(\mathbb{Z}_9)) = \{\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7},\bar{8}\}$ and the set of edge $E(TI(\mathbb{Z}_9)) = \{e_{\bar{2},\bar{4}}, e_{\bar{2},\bar{8}}, e_{\bar{4},\bar{8}}, e_{\bar{5},\bar{7}}, e_{\bar{5},\bar{8}}, e_{\bar{7},\bar{8}}\}$. Graph $TI(\mathbb{Z}_9)$ illustrated in the Figure 1.*

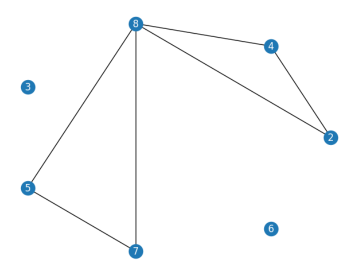


Fig. 1: Graph $TI(\mathbb{Z}_9)$

III. Result

In this section, the results of investigations regarding the conditions for connectivity of $TI(\mathbb{Z}_n)$ are given. The following is a theorem regarding the condition for the $TI(\mathbb{Z}_n)$ to be an empty graph.

Theorem 3. *Let $\mathbb{Z}_n$. If $n \leq 6$, then $TI(\mathbb{Z}_n)$ is an empty graph.*

Proof. The proof given by 3 cases below.

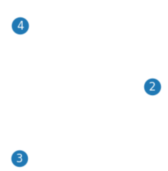
1) For $n = 3,4$.
Since $|V(TI(\mathbb{Z}_n))| < 3$, then there are not found any adjacency such that for $TI(\mathbb{Z}_n)$ has no edge or $TI(\mathbb{Z}_n)$ is an empty graph.
2) For $n = 5$.
There are found $V(TI(\mathbb{Z}_5)) = \{\bar{2},\bar{3},\bar{4}\}$ and $I(\mathbb{Z}_5) = \{\bar{0},\bar{1}\}$. Since $|V(TI(\mathbb{Z}_5))| = 3$ and $\mathbb{Z}_5$ is a commutative ring such that there is only one possible combination of vertices *u,v,w* i.e. $u = \bar{2}, v = \bar{3}, w = \bar{4}$. As a result of $uv = \bar{1}, \bar{1} \in I(\mathbb{Z}_5)$, then u,v,w are not adjacent. Therefore $TI(\mathbb{Z}_5)$ has no edge or $TI(\mathbb{Z}_5)$ is an empty graph.
3) For $n = 6$.
There are found $V(TI(\mathbb{Z}_6)) = \{\bar{2},\bar{3},\bar{4},\bar{5}\}$ and $I(\mathbb{Z}_6) = \{\bar{0},\bar{1},\bar{3},\bar{4}\}$. Since $|V(TI(\mathbb{Z}_6))| = 4$, same as before, then there are four possible combinations of vertices *u,v,w*. First, for vertex $u = \bar{2}, v = \bar{3}, w = \bar{4}$. As result of $uv = \bar{0}$ $\bar{0} \in I(\mathbb{Z}_6)$, then u,v,w are not adjacent. In the same way, for the others combination as well. Therefore, $TI(\mathbb{Z}_6)$ has no edge or $TI(\mathbb{Z}_6)$ is an empty graph.

So, it is proven that for $\mathbb{Z}_n$ where $n \leq 6$, the $TI(\mathbb{Z}_n)$ is an empty graph. ■

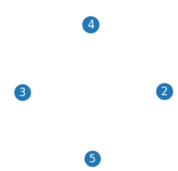
The illustrations of graph $TI(\mathbb{Z}_n)$ where $n \leq 6$ are shown in Figure 2.

(a) $TI(\mathbb{Z}_3)$

(b) $TI(\mathbb{Z}_4)$



(c) $TI(\mathbb{Z}_5)$



(d) $TI(\mathbb{Z}_6)$

Fig. 2: Graph $TI(\mathbb{Z}_n)$ where $n \leq 6$

There is a lemma that related to divisibility properties for any non-zero element in a field F.

Lemma 4. *Let F be a field. For every $a,b \in F$ where $a \neq 0$ and $b \neq 0$ then $a|b$ and $b|a$.*

Proof. Let non zero element $a,b \in F$. We will show that $a|b$ and $b|a$. Because of F is a field, clearly there exist b^{-1} such that $a = a.e = a.b.b^{-1}$. Using commutative and closed properties, then $a.b.b^{-1} = a.b^{-1}.b = c.b$ with $c = a.b^{-1}, c \in F$. Therefore, surely $b|a$. As the same way, for $a|b$. So, for every non zero element $a,b \in F$ then $a|b$ and $b|a$. ■

In the following, a lemma is given regarding cases of vertices that are not adjacent to each other in $TI(\mathbb{Z}_n)$.

Lemma 5. *Let $\mathbb{Z}_n$ where n is prime and $n \geq 7$. For every $u,v \in V(TI(\mathbb{Z}_n))$, u not adjacent to v if $uv = 1$ or $uv = x \neq 1$ where $x = u^{-1}$ or $x = v^{-1}$.*

Proof. Let $u,v \in V(TI(\mathbb{Z}_n))$ be distinct and arbitrary vertices. Then, clearly that $\mathbb{Z}_n$ where n prime are field, such that $I(\mathbb{Z}_n) = \{0,1\}$ where element 0 and 1 in $\mathbb{Z}_n$ are related to $\bar{0}$ and $\bar{1}$. Since, the field has no zero divisor element, then adjacency condition can be reduced to $uv \neq 1, vw \neq 1, uw \neq 1$ and $uvw = 1$. There will be showed 2 cases where u is not adjacent to v.

1) Case 1 $uv = 1$.
By Definition 1, if $uv = 1$, $1 \in I(\mathbb{Z}_n)$, then it will be contradiction with one of adjacency conditions of $TI(\mathbb{Z}_n)$. Therefore, u is not adjacent to v.
2) Case 2 $uv = x \neq 1$, where $x = u^{-1}$ or $x = v^{-1}$.
For $uv = u^{-1}$, if both side multiply with u, then $u.v.u = 1$. Seen that needed two elements of u in the left side so that triple vertices multiplication that involved u and v is equal to 1. By Definition 1, it will be contradiction with one of adjacency conditions where $u \neq v \neq w$ respectively. Therefore, u is not adjacent to v. In the same ways for $uv = v^{-1}$.

■

As an illustration of the lemma 5, the following is an example of an explanation of the cases that two vertices is not adjacent to each other in $\mathbb{Z}_{11}$.

Example 6. *Let $\mathbb{Z}_{11}$. The set of vertex element and the set of idempotent element, $V(TI(\mathbb{Z}_{11})) = \{\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7},\bar{8},\bar{9},\bar{10}\}$ and $I(TI(\mathbb{Z}_{11})) = \{\bar{0},\bar{1}\}$. We provide an example explanation by showing that any two vertices that are not adjacent in $\mathbb{Z}_{11}$, can be included in one of the two cases in the Lemma 5 above. Given below adjacency matrix of $\mathbb{Z}_{11}$ in the Table I.*

TABLE I: *Adjacency Matrix* of $TI(\mathbb{Z}_{11})$

	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$	$\bar{10}$
$\bar{2}$	0	0	1	1	0	1	1	1	1
$\bar{3}$	0	0	0	0	1	1	1	0	1
$\bar{4}$	1	0	0	0	0	1	1	0	1
$\bar{5}$	1	0	0	0	1	1	0	0	1
$\bar{6}$	0	1	0	1	0	1	1	1	1
$\bar{7}$	1	1	1	1	1	0	0	0	1
$\bar{8}$	1	1	1	0	1	0	0	1	1
$\bar{9}$	1	0	0	0	1	0	1	0	1
$\bar{10}$	1	1	1	1	1	1	1	1	0

Seen that $\bar{2}$ is not adjacent to $\bar{3}$. This is because $\bar{2}.\bar{3} = \bar{6} = \bar{2}^{-1}$ such that include in case 2. Then, vertex $\bar{2}$ is also not adjacent to $\bar{6}$ because $\bar{2}.\bar{6} = \bar{1}$ such that include in case 1. Now, vertex $\bar{3}$ is not adjacent to $\bar{4}$ because $\bar{3}.\bar{4} = \bar{1}$ such that include in case 1. Also, vertex $\bar{3}$ is not adjacent to $\bar{5}$ because $\bar{3}.\bar{5} = \bar{4} = \bar{3}^{-1}$ such that include in case 2. In the same ways, for the others vertices that not adjacent each other in $TI(\mathbb{Z}_{11})$ and always can be included to one of the two cases in Lemma 5. The $TI(\mathbb{Z}_{11})$ is showed in Figure 3.

The following result show that for $\mathbb{Z}_n$ where n prime and $n \geq 7$, $TI(\mathbb{Z}_n)$ is a connected graph.

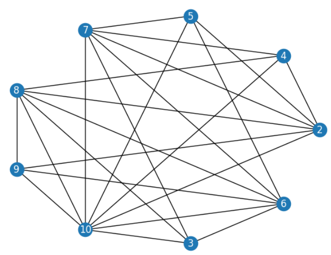


Fig. 3: Graph $TI(\mathbb{Z}_{11})$

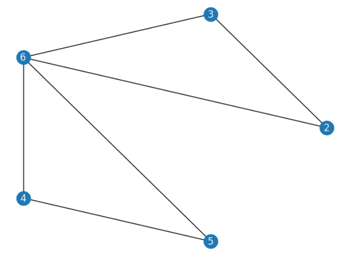


Fig. 4: $TI(\mathbb{Z}_7)$

Theorem 7. *Let $\mathbb{Z}_n$. If n prime and $n \geq 7$, then $TI(\mathbb{Z}_n)$ is a connected graph.*

Proof. Let $u, v \in V(TI(\mathbb{Z}_n))$ be arbitrary. Then, by Lemma 5 adjacency condition can be reduced to $uv \neq 1, vw \neq 1, uw \neq 1$ and $uvw = 1$. If u and v are adjacent, then there is a direct path $u - v$. If u and v are not adjacent, then by Lemma 5 there are divided to 2 cases.

1) Case 1. If $uv = 1$. Let $w \in V(TI(\mathbb{Z}_n))$. There exist $w^{-1} \in V(TI(\mathbb{Z}_n))$ where $w.w^{-1} = 1$. Since $u \in V(TI(\mathbb{Z}_n))$, then by Lemma 4, $w^{-1} = u.x$ such that $w.u.x = w.w^{-1} = 1$. Therefore, u adjacent to w. In the same way for v, such that v adjacent to w. Since u and v are adjacent to w, then there is a path u–w–v.
2) Case 2. If $uv = x \neq 1$, where $x = u^{-1}$ or $x = v^{-1}$. Let $uv = u^{-1}$. Since $u^{-1} \in V(TI(\mathbb{Z}_n))$, then by Lemma 4 there exist $w \in V(TI(\mathbb{Z}_n))$ such that $u^{-1} = w.y$. So, $u.w.y = u.u^{-1} = 1$ such that u adjacent to w. Then, also for $v^{-1} \in V(TI(\mathbb{Z}_n))$ by Lemma 4, v^{-1} can be divided by w such that $v^{-1} = w.z$. Therefore, $v.w.z = v.v^{-1} = 1$, then v adjacent to w. Since u adjacent to w and v adjacent to w then there is a path u–w–v. With same ways, can be proved for $uv = v^{-1}$.

Since there is always can be found a path between u and v, then $TI(\mathbb{Z}_n)$ where n prime and $n \geq 7$ is a connected graph. ■

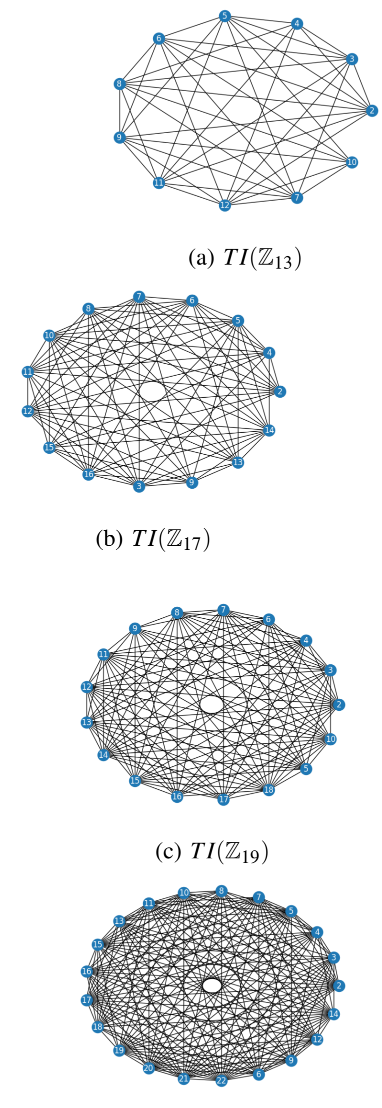
Given example that related to a connected graph of $TI(\mathbb{Z}_7)$ as an illustration of Theorem 7.

Example 8. *Let $\mathbb{Z}_7$. There is obtained vertex set $V(TI(\mathbb{Z}_7)) = \{2,3,4,5,6\}$ and idempotent set $I(\mathbb{Z}_7) = \{0,1\}$. Now, let $u, v \in V(TI(\mathbb{Z}_7))$ be distinct vertices, will be shown there always exist path between u and v. For $u = 2$ and $v = 3$, there exist $w = 6$ such that $u.v.w = \bar{1} \in I(\mathbb{Z}_7)$. Therefore, there is a direct path between $2 - 3$, $2 - 6$ and $3 - 6$.*

For $u = \bar{2}$ and $v = \bar{4}$, as result of $\bar{2}.\bar{4} = \bar{1}, \bar{1} \in I(\mathbb{Z}_7)$ then vertex $\bar{2}$ and $\bar{4}$ are not adjacent. So, there exist $w = \bar{6}$ such that $w^{-1} = \bar{6} = \bar{2}.\bar{3} = \bar{4}.\bar{5}$. Therefore, $\bar{2}$ and $\bar{4}$ adjacent to $\bar{6}$. Can be found a path $\bar{2} - \bar{6} - \bar{4}$, as the same ways for $u = \bar{3}$ and $v = \bar{5}$. For vertex $u = \bar{2}$ and $v = \bar{5}$, $u.v = \bar{3}$ where $\bar{3}$ is the inverse of vertex $\bar{5}$. There exist $w = \bar{6}$ such that $v^{-1} = \bar{3} = \bar{6}.\bar{4}$ and $u^{-1} = \bar{4} = \bar{3}.\bar{6}$. Then, clearly $\bar{2}$ and $\bar{5}$ adjacent to $\bar{6}$ such that there is a path $\bar{2} - \bar{6} - \bar{5}$ as the same ways for $u = \bar{3}$ and $v = \bar{4}$.

Since there is always found a path between every two distinct vertices, then $TI(\mathbb{Z}_7)$ is a connected graph. Figure 4 shows graph $TI(\mathbb{Z}_7)$.

As shown in Figure 3, $TI(\mathbb{Z}_{11})$ is a connected graph because it also satisfies Theorem 1. In addition, we also give several other graphs $TI(\mathbb{Z}_n)$ where n is prime and $n \geq 7$ in Figure 5.



(a) $TI(\mathbb{Z}_{13})$

(b) $TI(\mathbb{Z}_{17})$

(c) $TI(\mathbb{Z}_{19})$

(d) $TI(\mathbb{Z}_{23})$

Fig. 5: Graph $TI(\mathbb{Z}_n)$ where $n = \{13, 17, 19, 23\}$.

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