Bi-Edge Metric Dimension of Graphs

Rinurwati and Fadillah Dian Maharani

***Abstract*—Given a connected $G = (V(G), E(G))$ graph. The main problem in graph metric dimensions is calculating the metric dimensions and their characterization. In this research, a new dimension concept is introduced, namely a bi-edge metric dimension of graph which is a development of the concpet of bi-metric graphs with the innovation of bi-metric graph representations to become the bi-edge metric graph representations. In this case, what is meant by bi-edge metric and edge detour. If there is a set in G that causes every edge in G has a different bi-edge metric representation in G, then that set is called the bi-edge metric resolving set. The minimum cardinality of the bi-edge metric resolving set graphs is called the bi-edge metric dimension of G graph, denoted by $e\dim_b(G)$. The spesific purpose of this research is to apply the concept of bi-edge metric dimensions to special graphs, such as cycle, complete, star and path can be obtained.**

***Index Terms*—Bi-Metric Dimension, Edge Metric Dimension, Edge Detour Dimension, Bi-Edge Metric Dimension.**

I. Introduction

THE metric dimension of an edge is a sub-field of research in graph theory where discussion has become lively starting when an articel was written in 2018 entitled *Uniquely Identifying The Edge of a Graph: The Edge Metric Dimension.* [1]. In the article discussed about the edge resolving set. A set W is said to be an edge resolving set if each edge in G graph has a different edge representation of W. The metric representation of an edge in G graph is measured based on the distance of that edge to the points in the resolving set. The metric dimension of the edge is the minimum cardinality of the edge resolving set in G.

Research on edge metric dimensions is developed from two perspectives, namely from the concept and the graph, especially the graph of the result of operations. The development of the metric dimension concept related to this research is the edge metric concept itself [1], the edge detour dimension concept and the bi-metric dimension. In 2010, an article entitled *On Edge Detour Graphs* [2] a new idea was raised, namely the edge detour dimension. The development of the edge detour dimension is the determination of an edge detour representation of every edge e in G graph against the edge detour resolving set, which is in the form of a k-vector of the pair of the length of the longest path or detour between edge e and vertex $v_j, j \in \mathbb{N}$ in the edge detour resolving set.

On the other hand, in 2014 there is an article entitled *Bi-Metric Dimension of Graphs* [3] a new idea was raised too, namely the bi-metric dimension. The development of this bi-metric dimension is the determination of the representation of each vertex u in G graph against the resolving set, which is in the form of a k-vector of the pair of distances and detour between vertex u and $v_j, j \in \mathbb{N}$ in the resolving set. The concept of bi-metric dimension was recently introduced in [3], has only been implemented to obtain the bi-metric dimension of some special graphs, and for the resulting graph the bi-metric dimension has not yet been produced, and the development of the new bi-metric concept there is one [4], so not much research has been done.

Therefore, the bi-metric dimension has the potential to be further developed, both for the variant concept of bi-metric dimension and for the bi-metric dimension of the resulting graph operation. In this study a new variant of the bi-metric dimension concept was developed which combines two dimension concepts called the bi-edge metric dimension graph, by developing a new method, namely a mixture of methods subgraph and pattern recognition.

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II. Main Results

We start this section with the definition of the bi-edge metric dimension as follows. Given a connected G graph with a set of vertices $V(G)$ and a set of edges $E(G)$. An ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ is called a bi-edge metric resolving set of G if every edge in G has a different representation to W and $e = xy$, and $x, y \in V(G)$. The bi-edge metric representation of $e \in E(G)$ with respect to W is a k-tuples,

$$r_{bs}(e|W) = ((d(e,w_1), d(e,w_2), \ldots, d(e,w_k)), (\delta(e,w_1), \delta(e,w_2), \ldots, \delta(e,w_k)))$$

where $d(e,w_i)$ and $\delta(e,w_i)$ are the distance and detour between an edge e and a vertex w_i, respectively, for $i \in \{1, 2, \ldots, k\}$. A set W is called a bi-edge metric resolving set of G if every edge in G has different a bi-edge metric representation with respect to W. The bi-edge metric resolving set that has minimal cardinality is called the bi-edge metric basis, while the number of vertices on the bi-edge metric basis is called the bi-edge metric dimension of G graph which is denoted by $e\dim_b(G)$. In the following is an example of obtaining the bi-edge metric dimension of an umbrella graph $U_{6,5}$ consisting of a path of order six and a path of order five in Figure 1. The graph in Figure 1 is an umbrella graph with a vertex set $V(U_{6,5}) = \{u_i, v_j | i \in \{1, 2, \ldots, 6\}, j \in \{1, 2, \ldots, 5\}\}$ and an edge set $E(U_{6,5}) = \{e_i | i \in \{1, 2, \ldots, 15\}\}$. Select an ordered set $W \subseteq V(U_{6,5})$, with $W = \{u_1, u_2, u_3, u_4, u_5\}$, then the bi-edge metric representation is obtained as follows

$$r_{bs}(e_1|W) = ((0,0,1,2,2),(6,6,6,6,6)),$$
$$r_{bs}(e_2|W) = ((1,0,0,1,2),(6,6,6,6,6)),$$
$$r_{bs}(e_3|W) = ((2,1,0,0,1),(6,6,6,6,6)),$$
$$r_{bs}(e_4|W) = ((1,2,1,0,0),(6,6,6,6,6)),$$
$$r_{bs}(e_5|W) = ((2,2,2,1,0),(6,6,6,6,6)),$$

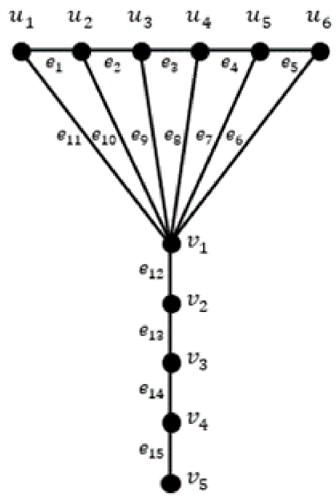

Fig. 1: $U_{6,5}$ Graph

$$\begin{aligned}
r_{bs}(e_6|W) &= ((1,1,1,1,1),(6,6,6,6,6)),\\
r_{bs}(e_7|W) &= ((1,1,1,1,0),(6,6,6,6,7)),\\
r_{bs}(e_8|W) &= ((1,1,1,0,1),(6,6,6,7,6)),\\
r_{bs}(e_9|W) &= ((1,1,0,1,1),(6,6,7,6,6)),\\
r_{bs}(e_{10}|W) &= ((1,0,1,1,1),(6,7,6,6,6)),\\
r_{bs}(e_{11}|W) &= ((0,1,1,1,1),(7,6,6,6,6)),\\
r_{bs}(e_{12}|W) &= ((1,1,1,1,1),(7,6,5,5,6)),\\
r_{bs}(e_{13}|W) &= ((2,2,2,2,2),(8,7,6,6,7)),\\
r_{bs}(e_{14}|W) &= ((3,3,3,3,3),(9,8,7,7,8)),\\
r_{bs}(e_{15}|W) &= ((4,4,4,4,4),(10,9,8,8,9)).
\end{aligned}$$

Because every edge in $U_{6,5}$ has different bi-edge metric representations with respect to W, then W is a bi-edge metric resolving set of $U_{6,5}$ graph. Furthermore, it is shown that W is a bi-edge metric resolving set with minimal cardinality. Take any $Q \subseteq V(U_{6,5})$ with $|Q| < |W|$ so $|Q| = 4$, then there are several cases that prove that Q is not a bi-edge metric resolving set of $U_{6,5}$ graph. It can be concluded that W is a bi-edge metric basis of $U_{6,5}$. So, $e_b\dim(U_{6,5}) = 5$. ■

Below are several lemmas that can help in proving theorems relating to bi-edge metric dimension.

Lemma 1. *Let G be a connected graph with order n and two ordered sets $A,B \subseteq V(G)$. If $A \subseteq B$ and A is a bi-edge metric resolving set of G graph, then B is also a bi-edge metric resolving set of G graph.*

Proof. Suppose $A = \{v_1,v_2,\ldots,v_l\}$ and $B = \{v_1,v_2,\ldots,v_m\}$ with $l \le m \le n$. It is known that A is a bi-edge metric resolving set of G graph. An ordered set A is called a bi-edge metric resolving set of G graph if every edge in G has different bi-edge metric representations with respect to A. Because $A \subseteq B$, then $A \cap B = \{v_1,v_2,\ldots,v_l\} \cap \{v_1,v_2,\ldots,v_m\} = \{v_1,v_2,\ldots,v_m\} = B$. As a result, all edges in G have a different bi-edge metric representations with respect to B. So B is a bi-edge metric resolving set of G graph. ■

Lemma 2. *Let G be a connected graph with order n and two ordered sets $A,B \subseteq V(G)$. If $B \supseteq A$ and B is not a bi-edge metric resolving set of G graph, then A is also not a bi-edge metric resolving set of G graph.*

Proof. It is known that B is not a bi-edge metric resolving set of G graph. So for every edge in G has at least two edges that have the same bi-edge metric representation with respect to B. Because $B \supseteq A$, then $A \cup B = \{v_1,v_2,\ldots,v_l\} \cup \{v_1,v_2,\ldots,v_m\} = \{v_1,v_2,\ldots,v_m\} = B$. It is clear that for every edge in G has at least two edges that have the same bi-edge metric representation with respect to A. So A is not a bi-edge metric resolving set of G graph ■

The main purpose of this research is to determine the exact values of a bi-edge metric dimension of C_n, K_n, S_n and P_n. In the following the theorem of a bi-edge metric dimension is presented.

Theorem 3. *Let G be a cycle with order $n \ge 3$, then $e\dim_b(G) = 2$.*

Proof. Let $V(G) = \{v_i | i \in \{1,2,3,\ldots,n\}\}$ with $E(G) = \{v_iv_{i+1} | i \in \{1,\ldots,n-1\}\} \cup \{v_nv_1\}$. Let $W = \{v_i, v_{i+1} | i \in \{1,2,\ldots,n\}\} \subseteq V(G)$ with $|W| = 2$. Without loss of generality, choose $W = \{v_1,v_2\}$, then there are two possible cases for n are presented below: (1) $n = 2k+1$ and (2) $n = 2k$ with $k \in \mathbb{N}$.

(1) When $n = 2k+1, k \in \mathbb{N}$, for every $v_iv_{i+1}, v_nv_1 \in E(G)$ with $i \in \{1,2,\ldots,n-1\}$, applies:

$$d(v_iv_{i+1},v_1) = \begin{cases} i-1, & i \in \{1,2,\ldots,\lceil \frac{n}{2} \rceil\} \\ n-i, & i \in \{\lceil \frac{n}{2} \rceil +1, \lceil \frac{n}{2} \rceil +2, \ldots, n-1\} \end{cases}$$
$$d(v_nv_1,v_1) = 0$$
$$d(v_iv_{i+1},v_2) = \begin{cases} i-1, & i \in \{1\} \\ i-2, & i \in \{2,3,\ldots,\lceil \frac{n}{2} \rceil +1\} \\ n-i, & i \in \{\lceil \frac{n}{2} \rceil +2, \lceil \frac{n}{2} \rceil +3, \ldots, n-1\} \end{cases}$$
$$d(v_nv_1,v_2) = 1$$

(2) When $n = 2k, k \in \mathbb{N}$, for every $v_iv_{i+1}, v_nv_1 \in E(G)$ with $i \in \{1,2,\ldots,n-1\}$, applies:

$$d(v_iv_{i+1},v_1) = \begin{cases} i-1, & i \in \{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\} \\ n-i, & i \in \{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil +1, \ldots, n-1\} \end{cases}$$
$$d(v_nv_1,v_1) = 1$$
$$d(v_iv_{i+1},v_2) = \begin{cases} i-1, & i \in \{1\} \\ i-2, & i \in \{2,3,\ldots,\lceil \frac{n}{2} \rceil\} \\ n+1-i, & i \in \{\lceil \frac{n}{2} \rceil +1, \lceil \frac{n}{2} \rceil +2, \ldots, n-1\} \end{cases}$$
$$d(v_nv_1,v_2) = 0$$

Because every edge in G has different representation with respect to W, then W is a bi-edge metric resolving set of G graph. Next, it is shown that W is a bi-edge metric resolving set with minimal cardinality. Take any $S \subseteq V(G)$ with $|S| < |W|$. Suppose $|S| = 1$, without loss of generality, choose $S = \{v_1\}$, then there are two edges $v_1v_2, v_nv_1 \in E(G)$ have the same representation with respect to S so that $r_{bs}(v_1v_2|S) = ((0),(n-1)) = r_{bs}(v_nv_1|S)$. Consequently, S is not a bi-edge metric resolving set of G graph. So, W is a bi-edge metric basis of G graph and $e\dim_b(G) = 2$. ■

The bi-edge metric dimension of K_n graph with order $n \ge 4$ is $n-1$. The proof of $e\dim_b(K_n) = n-1$ is contained in Theorem 4.

Theorem 4. *Let G be a complete with order $n \geq 4$, then $e\dim_b(G) = n-1$.*

Proof. Let $V(G) = \{v_i | i \in \{1,2,3,\ldots,n\}\}$ with $E(G) = \{v_iv_j | i,j \in \{1,\ldots,n\}; i \neq j\}$. Let $W = \{v_i | i \in \{1,2,\ldots,n\}\} \subseteq V(G)$ with $|W| = n-1$. Without loss of generality, choose $W = \{v_1, v_2, \ldots, v_{n-1}\}$. Because G is complete, then there are as many $n-1$ edges $v_nv_k \in E(G)$ with $k \in \{1,2,\ldots,n-1\}$ which does not contain any vertex in W so that a bi-edge metric representation in G is obtained.

$$r_{bs}(v_nv_1|W) = \left(\left(0, \underbrace{1,1,\ldots,1}_{n-2}\right), \left(n, \underbrace{n-1,n-1,\ldots,n-1}_{n-2}\right)\right)$$
$$r_{bs}(v_nv_2|W) = \left(\left(1,0, \underbrace{1,1,\ldots,1}_{n-3}\right), \left(n-1,n, \underbrace{n-1,n-1,\ldots,n-1}_{n-3}\right)\right)$$
$$\vdots$$
$$r_{bs}(v_nv_{n-2}|W) = \left(\left(\underbrace{1,1,\ldots,1}_{n-3},0,1\right), \left(\underbrace{n-1,n-1,\ldots,n-1}_{n-3},n,n-1\right)\right)$$
$$r_{bs}(v_nv_{n-1}|W) = \left(\left(\underbrace{1,1,\ldots,1}_{n-2},0\right), \left(\underbrace{n-1,n-1,\ldots,n-1}_{n-2},n\right)\right).$$

Because every edge in G has different representation with respect to W, then W is a bi-edge metric resolving set of G graph. Next, it is shown that W is a bi-edge metric resolving set with minimal cardinality. Take any $S \subseteq V(G)$ with $|S| < |W|$. Suppose $|S| = n-2$, without loss of generality, choose $S = \{v_1, v_2, \ldots, v_{n-2}\}$, then there are two edges $v_1v_{n-1}, v_1v_n \in E(G)$ have the same representation with respect to S so that

$$r_{bs}(v_1v_{n-1}|S) = \left(\left(0, \underbrace{1,1,\ldots,1}_{n-1},0\right), \left(\underbrace{n-1,n-1,\ldots,n-1}_{n-2}\right)\right)$$
$$= r_{bs}(v_1v_n|S),$$

$r_{bs}(v_1v_2|S) = \big((0),(n-1)\big) = r_{bs}(v_nv_1|S)$. Consequently, S is not a bi-edge metric resolving set of G graph and every $S' \subseteq V(G)$ with $|S'| < |S|$ is also not a bi-edge metric resolving set of G graph (Lemma 2). So, W is a bi-edge metric basis of G graph and $e\dim_b(G) = n-1$. ■

The bi-edge metric dimension of S_n graph with order $n \geq 4$ is $n-2$. The proof of $e\dim_b(S_n) = n-2$ is contained in Theorem 5.

Theorem 5. *Let G be a star with order $n \geq 4$, then $e\dim_b(G) = n-2$.*

Proof. Let $V(G) = \{v_i | i \in \{1,2,3,\ldots,n\}\}$ with $E(G) = \{v_1v_i | i \in \{2,3,\ldots,n\}\}$. Let $W = \{v_i | i \in \{2,3,\ldots,n\}\} \subseteq V(G)$ with $|W| = n-2$. Without loss of generality, choose $W = \{v_2, v_3, \ldots, v_{n-1}\}$. Because G is star, then there is an edge $v_1v_n \in E(G)$ which does not contain any vertex in W so that

$$r_{bs}(v_1v_n|W) = \left(\left(\underbrace{1,1,\ldots,1}_{n-2}\right), \left(\underbrace{2,2,\ldots,2}_{n-2}\right)\right).$$

Because every edge in G has different representation with respect to W, then W is a bi-edge metric resolving set of G graph. Next, it is shown that W is a bi-edge metric resolving set with minimal cardinality. Take any $S \subseteq V(G)$ with $|S| < |W|$. Suppose $|S| = n-3$, without loss of generality, choose $S = \{v_2, v_3, \ldots, v_{n-2}\}$, then there are two edges $v_1v_{n-1}, v_1v_n \in E(G)$ have the same representation with respect to S so that

$$r_{bs}(v_1v_{n-1}|S) = \left(\left(\underbrace{1,1,\ldots,1}_{n-3}\right), \left(\underbrace{2,2,\ldots,2}_{n-3}\right)\right) = r_{bs}(v_1v_n|S).$$

Consequently, S is not a bi-edge metric resolving set of G graph and every $S' \subseteq V(G)$ with $|S'| < |S|$ is also not a bi-edge metric resolving set of G graph (Lemma 2). So, W is a bi-edge metric basis of G graph and $e\dim_b(G) = n-2$. ■

The bi-edge metric dimension of P_n graph with order $n \geq 4$ is one. The proof of $e\dim_b(P_n) = 1$ is contained in Theorem 6.

Theorem 6. *Let G be a path with order $n \geq 4$, then $e\dim_b(G) = 1$.*

Proof. Let $V(G) = \{v_i | i \in \{1,2,3,\ldots,n\}\}$ with $E(G) = \{v_iv_{i+1} | i \in \{1,2,\ldots,n-1\}\}$. Choose $W = \{v_1\} \subseteq V(G)$, then for every $v_iv_{i+1} \in E(G)$ with $i \in \{1,2,\ldots,n-1\}$ applies $d(e_i, v_1) = i-1$. Because every edge in G has different representation with respect to W, then it is clear that W is a bi-edge metric basis of G graph and $e\dim_b(G) = 1$. ■

III. Conclusion

From this research, the exact values of a bi-edge metric dimension are obtained for several special graphs, namely cycle, complete, star and path. Based on the results that has been obtained, this research can be developed to determine the bi-edge metric dimension on graphs resulting such as graphs resulting from corona operations.

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