Modeling and Estimating GARCH-X and Realized GARCH Using ARWM and GRG Methods*

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***Abstract*—This study evaluates the fitting performance of GARCH-X(1,1) and RealGARCH(1,1) models, which are extensions of GARCH(1,1) model by adding the Realized Kernel measure as an exogenous component, on real data, namely the Financial Times Stock Exchange 100 and Hang Seng stock indices over the period from January 2000 to December 2017. The models assume that the return error follows Normal and Student-*t* distributions. The parameters of models are estimated by using the Adaptive Random Walk Metropolis (ARWM) method implemented in Matlab and the Generalized Reduced Gradient (GRG) method. The comparison of estimation results shows that the GRG method has a good ability to estimate the models because it provides the estimation results that are close to the results of the ARWM method in terms of relative error. On the basis of Akaike Information Criterion, the RealGARCH models perform better than the GARCH-X models, where the RealGARCH model with Student-*t* distribution provides the best fit.**

***Index Terms*—Adaptive, GARCH-X, GRG, Realized GARCH, Realized Kernel**

I. INTRODUCTION

VOLATILITY has an important role in the strategic economic decisions because it can be interpreted as the standard deviation of changes in financial asset returns within a given period of time [1]. When the volatility of financial assets is higher, the risk is also higher. Financial assets include, among others, exchange rates of currency, stock indeks, and commodities.

The volatility of a time series data can be heteroscedastic, which means that the volatility value varies over time. One of the popular models that can be used to model the time-varying volatility is Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model introduced in [2]. Conventional GARCH-type models utilize daily returns (typically squared values of market returns) to extract information about the volatility at the daily level in asset market. Since the squared returns are contamined by noise, see [3], the use of high frequency data (per second, minute, hour and so on) providing intraday observations have become an increasingly popular way to obtain less noisy volatility [4], known as Realized Volatility (RV).

Study in [5] extended the GARCH model to the GARCH-X by incorporating the RV data as an exogenous component in the conditional volatility process. The model has been empirically proven to have a significant advantage in fitting to real data. The GARCH-X model is then extended in [4] to the Realized GARCH (RealGARCH) by expressing the RV as a stochastic equation depending on several unknown parameters.

Furthermore, previous empirical studies have shown that time series financial return are not normally distributed but have a heavy-tails characteristic. One of the proposed distributions that able to accommodate the heavy-tails characteristic is Student-*t* distribution. Recently, this empirical study was conducted in [6] in the context of non-linear GARCH-X model, in [7] in the context of logarithmic RealGARCH (log-RealGARCH) model, and in [8] in the context of GJR model.

In estimating the GARCH-type model, the MLE (Maximum Likelihood Estimation) based method is commonly used. In contrast to this, this study applies the Generalized Reduced Gradient (GRG) method with the reason to provide an overview of the ease of estimation on the proposed models for financial practitioners. As a reference for the estimation accuracy of the GRG method, the estimation result of the Adaptive Random Walk Metropolis (ARWM) method is assumed as true value. The ARWM method has been successfully employed in [6], [7], [8] and they showed the efficiency of the method.

Motivated by the above studies, the first aim of this study is to evaluate the accuracy of the GRG method in estimating the GARCH-X and RealGARCH models by comparing their estimation results to the results by the ARWM method. It contributes to providing an explanation that it is possible to estimate volatility models by using the GRG method and illustrates its use through empirical data. The second aim is to investigate the fitting performance of the GARCH-X (1,1) and RealGARCH (1,1) models by assuming the return error is Normally and Student-*t* distributed. This contributes to the literature on the choice of appropriate models to capture the characteristics of financial returns, especially in terms of volatility and distribution. To the best of the authors' knowledge, there is no literature on those works. The empirical analysis is based on real data, namely the FTSE100 (Financial Times Stock Exchange 100) and HSI (Hang Seng index) over the daily period from January 2000 to December 2017.

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II. MODEL AND ESTIMATION METHOD

A. GARCH-X(1,1) and RealGARCH(1,1) Models

GARCH-X model was introduced in [5] by directly adding realized measures to the conditional volatility equation of the GARCH model as exogenous variable. The model has been shown to improve the conventional GARCH model in the measurement of volatility and the accuracy of the model prediction. The GARCH-X(1,1) model typically takes the following form:

$$R_t = \varepsilon_t, \quad \varepsilon_t \sim N(0,\sigma_t^2) \tag{1}$$

$$\sigma_t^2 = \omega + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma X_{t-1} \tag{2}$$

where $\omega > 0$, $0 \leq \alpha, \beta < 1$, $\gamma > 0$ to ensure the positivity of the conditional variance and $0 \leq \alpha + \beta < 1$ to ensure the stationarity of the conditional variance. Meanwhile, the exogenous component X_t denotes the RV constructed from high frequency intraday returns. In our empirical study we use the Realized Kernel (RK) in [9] that is shown to be unbiased and converges at a faster rate than other RV measures. The GARCH-X(1,1) model has symmetrical volatility response to return shocks, meaning that conditional volatility is not determined by positivity or negativity unanticipated excess return [10].

The GARCH-X model was then developed in [4] to the RealGARCH model by expressing realized measure as an equation relating the observed realized measure to the latent volatility:

$$R_t = \varepsilon_t, \quad \varepsilon_t \sim N(0,\sigma_t^2) \tag{3}$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma X_{t-1} \tag{4}$$

$$X_t = \xi + \varphi \sigma_t^2 + \tau(z_t) + u_t, \quad u_t \sim N(0, s_u^2). \tag{5}$$

In this case, it should expect $0 < \varphi < 1$. To guarantee the conditional variance is finite and positive, the required conditions are, [11]:

$$\omega, \beta, \gamma, \omega, \varphi, s_u^2 > 0, \omega + \gamma\xi > 0, 0 < \beta + \gamma\varphi < 1. \tag{6}$$

The model facilitates a leverage effect that is denoted by the function $\tau(z_t) = \tau_1 z_t + \tau_2(z_t^2 - 1)$, where $z_t = \frac{R_{t-1}}{\sigma_{t-1}}$ and τ_1, $\tau_2 \in \mathbb{R}$. The function can generate volatility asymmetry in response to return shocks, which is indicated by a negative value of τ_1. Negative asymmetry effects in volatility interprets that volatility tends to be higher in response to 'bad news' (excess returns lower than expected) than to 'good news' and to be lower in response to 'good news' (excess returns higher than expected) than to 'bad news' [12].

B. Normal and Student-t Distributions

The Normal distribution, also called the Gauss distribution, is the most common type of distribution used in various statistical analyses. The curve of the Normal distribution is a symmetrical bell that extends infinitely in both positive and negative directions. However, not all symmetrical bell-shaped distributions are normal, for example the Student-*t* distribution. This distribution is bell-shaped symmetrical but has a thicker tail (often called heavy/fat tails) than the Normal distribution.

When the return error, ε_t, and the RV error, s_u, is standard Normally distributed, the log-likelihood functions corresponding to return and measure equations are:

$$\mathscr{L}(R_t|\omega,\alpha,\beta,\gamma) = -\frac{1}{2}\left[\log\left(2\pi\sigma_t^2\right) + \frac{R_t^2}{\sigma_t^2}\right], \tag{7}$$

$$\mathscr{L}\left(X_t|\xi,\varphi,\tau_1,\tau_2,s_u^2\right) = -\frac{1}{2}\left[\log\left(2\pi s_u^2\right) + \frac{\left(X_t - \xi - \varphi\sigma_t^2 - \tau(z_t)\right)^2}{s_u^2}\right], \tag{8}$$

respectively. Meanwhile, when ε_t follows Student-*t* distribution with degrees of freedom $\nu > 2$ (which controls the thickness of the distribution tail, the log-likelihood function corresponding to return is as follows [13]:

$$\mathscr{L}(R_t|\omega,\alpha,\beta,\gamma,\nu) = \log\Gamma\left(\frac{\nu+1}{2}\right) - \log\Gamma\left(\frac{\nu}{2}\right) - \log\left(\sigma_t^2(\nu-2)\right) - \frac{1}{2}(\nu+1)\log\left(1 + \frac{R_t^2}{\sigma_t^2(\nu-2)}\right). \tag{9}$$

In particular, the log-likelihood function in the RealGARCH(1,1) model is the sum of the log-likelihood functions of return and realized measure.

C. Estimation Methods

This study particularly chooses the GRG method as a method to estimate the model parameters that maximize log-likelihood. The GRG method is based on work published in [14], [15]. The GRG method is a simple estimation method and is often used to solve optimization problems with non-linear objective and constraint functions. Three primary parameters form the basis of this method: objective function, decision variables, and constraints [16]. The non-linear problem of the form given by constraints inequality is solved by the addition of slack variables. It is essentially an expansion of the Simplex approach, sometimes referred to as a linear programming solver, which partitions the variables into the basic variables and the non-basic variables [17].

The fundamental idea of the GRG method is to use constraint equations to express basic variables in the form of non-basic variables. The objective function is then expressed in terms of non-basic variables only. The GRG method solves the original problem with a sequence of problems, each of which uses a linear approximation of its constraints. In each iteration, the linearization of the constraints is recalculated at the points found from the previous iteration. Typically, although the constraints are only approximate, the sub-problems produce points that are increasingly close to the optimal point. The nature of linearization is that, at the optimal point, the linearized problem has the same solution as the original problem.

In the GRG framework, initial values of the decision variables are considered as the initial solution and small changes in the initial values are expected to improve the parameter values and the objective function. When the problem is to maximize the objective function, this function value will gradually "increase" and when the problem is to minimize the objective function, this function value will gradually "decrease". Occasionally, when the objective function is changing very little between trial solutions or for other reasons, the solver will quit before reaching a locally optimal solution. When the message "Solver found a solution" displays, it indicates that no other set of choice variable values around the current values produces a better value for the objective function, indicating that the GRG method has discovered a locally optimal solution.

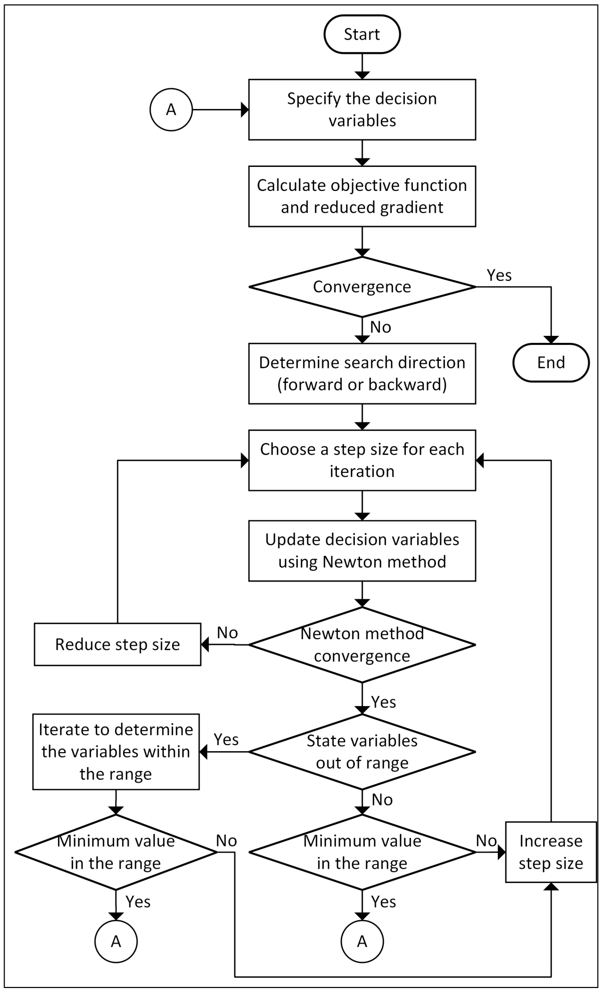

Fig. 1. Flowchart of GRG optimization algorithm [14], [18].

A flow chart of the GRG algorithm is shown in Figure 1. The following is a summary for each step [14], [19]:

1) **Specification of Decision Variables**: The algorithm begins with an initial solution guess of the decision variable values.
2) **Calculation of the Reduced Gradient**: The reduced gradient of the objective function is calculated with respect to the decision variables.
3) **Check for Convergence**: If the convergence criteria are satisfied, stop. Otherwise go to step 4.
4) **Compute the Search Direction**: A search direction is determined based on the reduced gradient, ensuring it improves the objective function while remaining within the feasible region defined by the constraints.
5) **Step Size Determination**: A line search is conducted to find the optimal step size along the search direction that optimizes the objective function.
6) **Iteration**: Steps 3–5 (2–5 if the reduced gradient is not computed in the one dimensional search) are repeated iteratively, with the algorithm updating the variable values and refining the search direction in each iteration.
7) **Termination**: The algorithm stops when the objective function or the variable values change by less than a specified tolerance, or when a maximum number of iterations is reached.

Studies in [20], [21] found that the GRG method is sensitive to the initial value, meaning that the different initial values for an unknown paramater may provide significantly different estimation results or may not even be found. Therefore, they suggested taking an initial value that is close to the expected optimum value.

In order to evaluate the accuracy of estimation results obtained by the GRG method. The ARWM method is used for comparison since the method was empirically shown in [6], [7], [22], to estimate GARCH(1,1)-type and logRealGARCH-type models efficiently in term of autocorrelation. The method is employed in the Markov Chain Monte Carlo (MCMC) scheme and implemented in Matlab by making own code. In contrast to the outcome of GRG method, some statistics such standard deviation and confidence intervals for the unknown parameters can be obtained by MCMC.

The steps of the ARWM method can be summarized as follows. Let θ be some unknown parameter of interest. At the i-th iteration, a proposal θ is generated by:

$$\theta^{(i)} = \theta^{(i-1)} + \sqrt{\Delta^{(i)}}z^{(i)}, \quad z^{(i)} \sim N(0,1), \tag{10}$$

where $\Delta^{(i)}$ is the step width. On the basis of the Bayesian approach, the posterior distribution of θ given data is calculated as follows:

$$\log p(\theta|\text{data}) = \mathscr{L}(\text{data}|\theta) + \log p(\theta), \tag{11}$$

where $p(\theta)$ is the prior distribution for θ. The proposal $\theta^{(i)}$ is accepted if $\frac{p\left(\theta^{(i)}|\text{data}\right)}{p\left(\theta^{(i-1)}|\text{data}\right)} > u$, in which $u \sim U(0,1)$.

After discarding the first N iteration, the remaining M samples are then used to calculated some statistics. Following the approach in [23], this study estimates the 95% Highest Posterior Density (HPD) interval as follows:

1) Calculate $M_{cut} = [0.05 \times M]$ and $M_{span} = M - M_{cut}$, where $[x]$ represents the standard rounding function of x.
2) Sort the estimated values from the smallest to the largest, i.e. $\left\{\theta_j\right\}_{j=1}^{M}$, where $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_M$.
3) Find the index j^* so that $\theta_{j^*+M_{span}} - \theta_{j*} = \min_{1 \leq j \leq M_{cut}} \left(\theta_{j+M_{span}} - \theta_j\right)$.
4) Determine the 95% HPD interval:

$$\left(\theta_{j^*}, \theta_{j^*+M_{span}}\right). \tag{12}$$

D. Evaluation of Model

In selecting the model that gives the best fit to real data, this study uses the Akaike Information Criterion (AIC). This method is used because the competing models do not need to be nested, meaning that neither model can be obtained as an appropriate parametric restriction on the other model. AIC's simplicity in calculation and application—it just needs the likelihood function and the model's parameter number—is one of its key benefits. AIC is independent of any external information, including the sample size and error distribution, which makes it most popular, flexible and reliable than other criteria [24]. AIC has a statistical value as follows:

$$\mathrm{AIC} = 2(k - \mathscr{L}), \tag{13}$$

where k represents the number of parameters in the estimated model and $\mathscr{L}$ represents the log-likelihood value of the model. The criterion is that the model with the smallest AIC value gives the best fit.

E. Steps Involved in Modeling

A flowchart of the GARCH-type methodology is shown in Figure 2. The steps involved in actually modeling a GARCH-type model are summarized as follows.

1) Data preparation
The preparation of the data is the initial step in GARCH modeling. This means collecting the historical information about the asset such as daily returns R_t, which are the natural logarithm of percentage changes in the asset value between two consecutive days, and daily Realized Kernel (RK) in [9] as the exogenous variable X_t. If we let P_t denote the value on day t and P_{t-1} denote the value on day t–1, then R_t can be expressed as

$$R_t = 100 \log\left(\frac{P_t}{P_{t-1}}\right). \tag{14}$$

To provide an empirical example, the observed data are two major stock market indices: FTSE100 (UK) and HSI (Hongkong) using a sample of daily returns and RK from January 2000 to December 2017, which are provided by the Oxford-Man Institute's "realised library".
2) Model specification
This study uses GARCH-X and RealGARCH with order (1,1), which has shown in [25], [26] in the GARCH(1,1) context to produce a simple and relatively accurate result for volatility estimate across a variety of fields. For the return errors, the standard Normal and Student-t distribution are introduced to capture various aspects of the returns process.
3) Define the log-likelihood function
For the GARCH-X model, the total log-likelihood function can be written as the sum of the log-likelihood function of the observed returns. For the RealGARCH model, the total log-likelihood function can be written as the sum of the log-likelihood functions of the observed returns and observed realized measures.
4) Estimation
While MLE is popular, other methods like GRG or Bayesian estimation can also be used. The optimization problem involves estimating the model parameter by maximizing the log-likelihood function. In comparison, the ARWM method in the Bayesian Markov Chain Monte Carlo (MCMC) algorithm is also performed.
5) Model diagnostics
From the results of the parameter estimation, the estimation accuracy of the GRG method is assessed against the estimation of the ARWM method as a respected benchmark. Meanwhile, the AIC assessments are used to compare competing models and to select the best-fitting one.

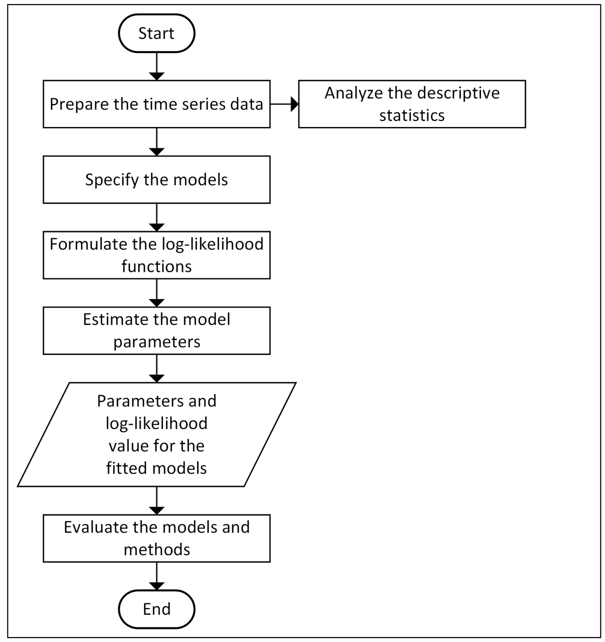

Fig. 2. The conceptual framework of GARCH modeling.

III. RESULT AND DISCUSSION

A. Description of Data

Table I presents summary statistics for the percentage return and RK of the FTSE100 and HSI data, in which SD denotes standard deviation. The means for both return data are close to zero as is expected for a time series. There is a high positive value of kurtosis (> 3), suggesting a heavy-tailed distribution for the return error. Although not reported, the Jarque–Bera normality test (see [27]) rejects the null hypothesis for both returns, which confirms non-normality of the series. Therefore, the assumption of Student-t distribution would be much appropriate in comparison to Normal distribution assumption.

TABLE I
DESCRIPTIVE STATISTICS.

Statistics	FTSE100		HSI	
	Return	RK	Return	RK
Mean	−0.035	0.766	−0.047	0.815
Standard Deviation	0.930	0.484	0.994	0.433
Maximum	7.04	5.71	12.16	6.68
Minimum	−5.76	0.20	−11.62	0.21
Kurtosis	7.53	15.20	16.13	31.72

Figure 3 displays the daily returns of FTSE100 and HSI. It can be seen that both returns fluctuates around their return means or it can be said that there is no upward or downward trend. This stationarity is a pre-condition before applying the GARCH model. On both plots of returns, the volatility of stock return may looks decrease after the middle of 2003 and then becomes much more volatile from 2008 to 2010. After 2010, the volatility tends to be smaller, with some turbulences in the middle of 2011 and the beginning of 2016.

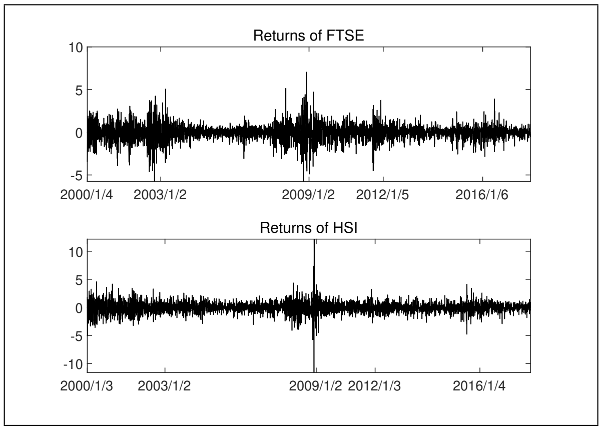

Fig. 3. Plots of daily return (in percentage).

B. Implementation of the Estimation Method

In implementing the ARWM method, the initial values for parameters in the GARCH-X(1,1) models are as follows:

$$\omega = 0.1, \alpha = 0.1, \beta = 0.9, \gamma = 0.1, \nu = 10, \quad (15)$$

in the RealGARCH(1,1) models are as follows:

$$\omega = 0.01, \beta = 0.7, \gamma = 0.2, \nu = 10, \quad (16)$$

$$\xi = 0.01, \varphi = 0.9, \tau_1 = 0.25, \tau_2 = -0.25, s_u^2 = 0.25. \quad (17)$$

Following the Bayesian MCMC rules, the model is completed through prior distributions for the model parameters. In this study, the prior distribution for the parameters ω, α, β, γ, ξ, φ, τ_1, τ_2, s_u^2 are $N(0,1000)$ as in [28] and for the parameter ν is $\exp(0.01)$ as in [29].

Since the GRG method is sensitive to the initial value, the initial values for the model parameters in the GRG method are taken to close the estimation results of the ARWM method:

$$\omega = 0.001, \alpha = 0.05, \beta = 0.9, \gamma = 0.05, \nu = 10 \quad (18)$$

for the GARCH-X(1,1) models, and

$$\omega = 0.001, \beta = 0.5, \gamma = 0.5, \nu = 10, \quad (19)$$

$$\xi = 0.1, \varphi = 0.9, \tau_1 = -0.05, \tau_2 = 0.05, s_u^2 = 0.05 \quad (20)$$

for the RealGARCH(1,1) models.

C. Efficiency of the ARWM Method

For example, Figure 4 provides trace plots of the posterior estimates for each parameter in the RealGARCHt(1,1) model adopting the FTSE100 stock index data. MCMC algorithm was run for a total of 6000 iterations with the first 1000 iterations discarded as burn-in period and the last 5000 iterations are used for inference. The trace plots seem to be stationary although the Markov chain explores the parameter space slowly for β, γ and φ. The Markov chains were then checked for convergence by integrated autocorrelation time of Sokal (1997) and they have autocorrelation time less than 180. It is reasonable to believe that the generated Markov chains have a good convergence and can be used as a reference for the accuracy of the accuracy of the estimation results by GRG method.

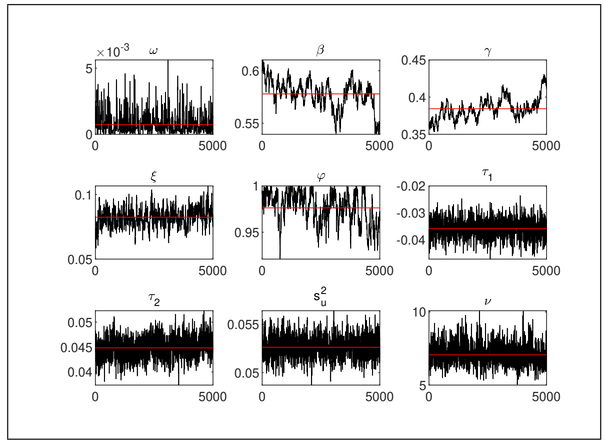

Fig. 4. Trace plots of the parameters of the RealGARCHt(1,1) model by using ARWM method for the FTSE100 data. Red line denotes the posterior mean.

D. Estimation Results

The results obtained of the GARCH-X(1,1) and RealGARCH(1,1) parameter estimation using the GRG and ARWM methods for the FTSE100 and HSI data sets are presented in Tables II and III, respectively. Assuming the estimated values obtained by the ARWM method as the actual measurement, the accuracy of the GRG method is measured by the relative error defined as the ratio of the absolute error of the measurement (difference between the measured value and the actual value) to the absolute of actual measurement. The formula is:

$$RE = \left| \frac{\text{actual value} - \text{measured value}}{\text{actual value}} \right| \quad (21)$$

Comparing between the estimated values of two methods, the results with both methods seem to be close to each other, with the exception of the parameter ω, which is zero by the GRG method. By ignoring ω, both methods give the estimation results that are not much different with the relative errors less than 10%. This indicates that the GRG method has a good ability to estimate the studied models. Notice that the constraint violation on parameter ω is caused by the estimated value which is very close to zero. This is similar to the results in [1], [20]. However, the violation does not affect the estimation of the other parameters.

Regarding the estimates of β, the GARCH-X models exhibit a high degree of volatility persistence about 0.9, which is greater than the degree of volatility persistence about 0.6 exhibited by the RealGARCH models. This finding suggests that changes in volatility of models with Student-t distribution have only a relatively small effect on asset prices. Meanwhile,

TABLE II
THE ESTIMATES FOR POSTERIOR MEAN IN ADOPTING THE FTSE100 DATA.

Par.	Normal		Student-t	
	GARCH-X	RealGARCH	GARCH-X	RealGARCH
Method: ARWM				
ω	0.0006	0.0006	0.0006	0.0007
α	0.106	-	0.103	-
β	0.838	0.562	0.850	0.578
γ	0.052	0.396	0.045	0.384
ν	-	-	9.69	7.07
τ_1	-	-0.036	-	-0.036
τ_2	-	0.044	-	0.045
ξ	-	0.084	-	0.082
φ	-	0.983	-	0.976
s_u^2	-	0.053	-	0.053
Method: GRG				
ω	0.000	0.000	0.000	0.000
α	0.104	-	0.099	-
β	0.840	0.565	0.856	0.563
γ	0.052	0.386	0.043	0.395
ν	-	-	9.43	6.46
τ_1	-	-0.035	-	-0.023
τ_2	-	0.044	-	0.046
ξ	-	0.084	-	0.083
φ	-	0.999	-	0.985
s_u^2	-	0.053	-	0.052
Relative Error (RE) in percent				
ω	-	-	-	-
α	1.06	-	3.56	-
β	0.23	0.53	0.69	1.50
γ	0.34	2.32	3.68	0.50
ν	-	-	2.66	1.52
τ_1	-	1.52	-	0.76
τ_2	-	1.17	-	1.74
ξ	-	0.54	-	1.06
φ	-	1.75	-	1.55
s_u^2	-	0.41	-	0.15

TABLE III
THE ESTIMATES FOR POSTERIOR MEAN IN ADOPTING THE HSI DATA.

Par.	Normal		Student-t	
	GARCH-X	RealGARCH	GARCH-X	RealGARCH
Method: ARWM				
ω	0.001	0.004	0.001	0.007
α	0.051	-	0.047	-
β	0.904	0.672	0.915	0.684
γ	0.045	0.323	0.038	0.314
ν	-	-	8.06	6.99
τ_1	-	-0.027	-	-0.027
τ_2	-	0.072	-	0.073
ξ	-	0.130	-	0.119
φ	-	0.849	-	0.839
s_u^2	-	0.050	-	0.050
Method: GRG				
ω	0.000	0.000	0.000	0.005
α	0.049	-	0.043	-
β	0.907	0.671	0.923	0.677
γ	0.044	0.331	0.034	0.319
ν	-	-	7.87	6.91
τ_1	-	-0.027	-	-0.027
τ_2	-	0.072	-	0.072
ξ	-	0.134	-	0.122
φ	-	0.828	-	0.841
s_u^2	-	0.050	-	0.050
Relative Error (RE) in percent				
ω	-	-	-	-
α	3.90	-	7.37	-
β	0.52	0.06	0.84	0.97
γ	3.30	2.41	9.05	1.83
ν	-	-	2.33	1.05
τ_1	-	0.39	-	0.19
τ_2	-	1.01	-	0.96
ξ	-	3.19	-	2.04
φ	-	2.53	-	0.27
s_u^2	-	0.03	-	0.18

the point estimate of φ suggests that volatilities of FTSE100 and HSI amount to about 99% and 84% of daily volatility, respectively. Observing the coefficient on z_t, that is τ_1, the estimate is negative in all cases. In terms of 95% HPD interval, τ_1 is significant in all cases with the estimate values between -0.05 and -0.02 on adopting the FTSE100 data and between -0.04 and -0.01 on adopting HSI data. The fact that this finding suggests a negative asymmetry effect on volatility. Finally, the models with Student-t distribution produces a small degrees of freedom, suggesting the existence of heavy-tails in the return distribution.

E. Model selection

In this study, the fitting performance of models is investigated in the sample data. Table IV presents the AIC values of competing models estimated by both methods. As the AIC for the model with Student-t distribution is smaller than the other distribution, it means the Student-t distribution is more adequate than the Normal distribution, confirming the previous result on the estimation of the degrees of freedom. Moreover, not surprisingly, the RealGARCH(1,1) model outperforms the GARCH-X(1,1) model in the case of each distribution. Therefore, overall, the RealGARCH(1,1) model with Student-t distribution provides the best data fit. In Figure 5 plots of time-series for daily conditional variance on the best model are depicted.

TABLE IV
THE AIC TEST FOR THE GARCH-X AND REALGARCH MODELS.

Model	Dist.	AIC		Ranking	
		ARWM	GRG	ARWM	GRG
Data: FTSE100					
GARCH-X	Normal	10230.7	10225.6	4	4
	Student-t	10152.8	10144.7	3	3
RealGARCH	Normal	10001.0	9993.6	2	2
	Student-t	9873.9	9881.7	1	1
Data: HIS					
GARCH-X	Normal	10547.3	10542.3	4	4
	Student-t	10444.2	10435.8	3	3
RealGARCH	Normal	10028.9	10020.5	2	2
	Student-t	9875.0	9864.4	1	1

IV. CONCLUSIONS

This study evaluated the modeling of GARCH-X(1,1) and RealGARCH(1,1) with Normal and Student-t distributions for return error based on the measure of Realized Kernel using the FTSE100 and HSI data. The models were estimated by using ARWM and GRG methods. The GRG method was demonstrated to have a good ability to estimate the studied models as the estimation results are not much different from

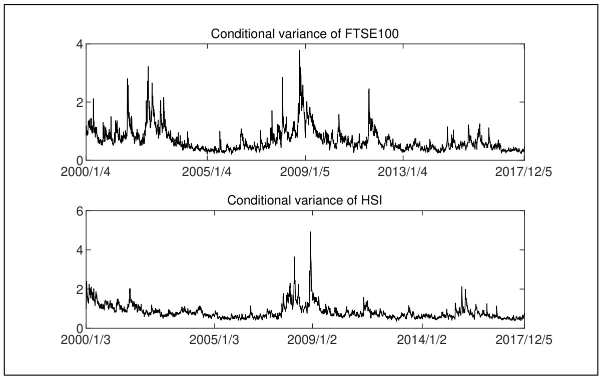

Fig. 5. Plots of daily conditional variance on the RealGARCHt(1,1) model.

the estimation results by the ARWM method, namely with a relative error of less than 2%.

The fitting performance of models was investigated by the AIC. We demonstrate that the models with the Student-t distribution consistently outperform those with the Normal distribution in all cases. Moreover, the use of RealGARCH(1,1) model instead of GARCH-X(1,1) model provides a better data fit for the conditional variance. Therefore, the RealGARCH(1,1) with Student-t distribution for return error yields the best fitting performance for both data.

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