Second Degree Refinement Jacobi Iteration Method for Solving System of Linear Equation

Tesfaye Kebede

***Abstract*—Several iterative techniques for the solution of linear system of equations have been proposed in different literature in the past.In this paper, we present a Second degree of refinement Jacobi Iteration method for solving system of linear equation, $Ax = b$ and we consider few numerical examples and spectral radius to show that the effective of the Second degree of refinement Jacobi Iteration Method (SDRJ) in comparison with other methods of First degree Jacobi (FDJ), First degree Refinement Jacobi (FDRJ) and Second degree Jacobi (SDJ) method.**

***Index Terms*—Jacobi iteration, second degree refinement, system of linear equations.**

I. Introduction

In many scientific and engineering applications, one often comes across with a problem of finding the solution of a system of linear equations written as the following equation in matrix form:

$$Ax = b \tag{1}$$

where A is a nonsingular matrix of size $n \times n$, x and b are n-dimensional vectors. Splitting the matrix A [1] as :

$$A = D - L - U \tag{2}$$

where D is a diagonal matrix and $-L$ and $-U$ are strictly lower and upper triangular part of A respectively. A general first degree linear stationary iterative method for the solution of the system of equation (1) may be defined in the form:

$$x^{(n+1)} = Hx^{(n)} + C \tag{3}$$

where $x^{(n+1)}$ and $x^{(n)}$ are the approximation for x at the $(n+1)^{th}$ and n^{th} iterations respectively, H is called the iterative matrix depending on matrix A and C is a column vector. The iteration system $x^{(n+1)} = Hx^{(n)} + C$ is converge if and only if the spectral radius of H are less than unity, i.e. $\sigma(H) < 1$.

The first degree iterative method of Jacobi (FDJ) method for the solution of (1) is defined as:

$$x^{(n+1)} = D^{-1}(L+U)x^{(n)} + D^{-1}b \tag{4}$$

and the first degree refinement Jacobi (FDRJ) method can be obtained in the form of :

$$x^{(n+1)} = (D^{-1}(L+U))^2x^{(n)} + (I + D^{-1}(L+U))D^{-1}b \tag{5}$$

$$X^{(n+1)} = H_{RJ}X^{(n)} + C_{RJ}, \tag{6}$$

where

$$H_{RJ} = [D^{-1}(L+U)]^2, C_{RJ} = [I + D - 1(L+U)]D^{-1}b \tag{7}$$

Manuscript received December 1, 2016; accepted February 11, 2017.

The author is with the Department of Mathematics, College of Science, Bahir Dar University, Bahir Dar, Ethiopia. Email: tk_ke@yahoo.com

The linear stationary any second degree method is given by [2]

$$x^{(n+1)} = x^{(n)} + a(x^{(n)} - x^{(n-1)}) + b_1(x^{(n+1)} - x^{(n)}) \tag{8}$$

Here $x^{(n+1)}$ appearing in the right hand side as given in equation (3) is completely consistent for any constant a and b_1 such that $b_1 \neq 0$.

$$X^{(n+1)} = X^{(n)} + a(X^{(n)} - X^{(n-1)}) + b_1(HX^{(n)} + C - X^{(n)})$$
$$\Rightarrow X^{(n+1)} = X^{(n)} + aX^{(n)} - aX^{(n-1)} + b_1HX^{(n)} + b_1C - b_1X^{(n)}$$
$$\Rightarrow X^{(n+1)} = X^{(n)} + aX^{(n)} - b_1X^{(n)} + b_1HX^{(n)} + b_1C - aX^{(n-1)}$$
$$\Rightarrow X^{(n+1)} = (1 + a - b1)X^{(n)} + b_1HX^{(n)} - aX^{(n-1)} + b_1C$$
$$\Rightarrow X^{(n+1)} = [(1 + a - b_1)I + b_1H]X^{(n)} - aX^{(n-1)} + b_1C$$

$$X^{(n+1)} = GX^{(n)} + H_1X^{(n-1)} + K \tag{9}$$

where

$$G = (1 + a - b_1)I + b_1H \tag{10}$$

$$H_1 = -aI \tag{11}$$

$$K = b_1C \tag{12}$$

The linear stationary any second degree method is given by [2] can be written in number (6) or number (7) with (8), (9) and (10) conditions. On the other way equation (1) can be solved using the second degree Jacobi(SDJ) stationary iterative method using

$$x^{(n+1)} = b_1D^{-1}(L+U)x^{(n)} - ax^{(n-1)} + b_1k_1$$
$$\Rightarrow x^{(n+1)} = b_1[D^{-1}(L+U)x^{(n)} + k_1] - ax^{(n-1)} \tag{13}$$

For optimal values of a and b_1 Where $k_1 = D^{-1}b$. If A is a row strictly diagonal dominant (SDD) matrix, then the Jacobi method converges for any arbitrary choice of the initial approximation [3].

In this paper, we construct a new method of solving a linear system of the form $Ax = b$ that arise in any engineering and applied science.

The outline of this paper is as follows: we introduce second degree refinement Jacobi (SDRJ) iterative method in accordance this we will see the relationship between spectral radius of first degree Jacobi (FDJ), first degree refinement Jacobi (FDRJ), Second degree Jacobi (SDJ) methods and Second degree refinement Jacobi iteration (SDRJ) methods are given. Based on the methods and results, few numerical examples are considered to show that the efficiency of the new method in comparison with the existing FDJ, FDRJ and SDJ methods. Finally discussion and conclusion made at Section V.

II. Second degree Refinement Jacobi (SDRJ) Iterative Method

Theorem 1: If matrix A is non singular PD and SDD matrix with $A = D - L - U$, then the Second degree of Refinement Jacobi iterative method is:

$$[x^{(n+1)} = b_1[D^{-1}(L+U)]^2 x^{(n)} - aIx^{(n-1)} + b_1(I + D^{-1}(L+U))D^{-1}b]$$

for any initial guess and the optimal values for a and b_1.

Given: A is non singular PD and SDD matrix and A = D-L-U. Required: the second degree of refinement Jacobi iterative method is: $x^{(n+1)} = b_1[D^{-1}(L+U)+k_1]x^{(n)} - a_1 x^{(n-1)}$. Proof: now consider equation (5)and (6), so one can get :

$$x^{(n+1)} = x^{(n)} + a(x^{(n)} - x^{(n-1)}) + b_1(H_{RJ}x^{(n)} + C_{RJ} - x^{(n)}) \quad (14)$$

This also can be written as follows after some computation:

$$X^{(n+1)} = G_{RJ}x^{(n)} + F_{RJ}x^{(n-1)} + K_{RJ} \quad (15)$$

where $G_{RJ} = (1+a-b_1)I + b_1 H_{RJ}$,$F_{RJ} = -aI$ and $K_{RJ} = b_1 C_{RJ}$. By using Golub and Varga [2]

$$\begin{pmatrix} x^{(n)} \\ x^{(n+1)} \end{pmatrix} = \begin{pmatrix} 0 & I \\ F_{RJ} & G_{RJ} \end{pmatrix} \begin{pmatrix} x^{(n-1)} \\ x^{(n)} \end{pmatrix} + \begin{pmatrix} 0 \\ K_{RJ} \end{pmatrix} \quad (16)$$

$$\Rightarrow \begin{pmatrix} x^{(n)} \\ x^{(n+1)} \end{pmatrix} = \widehat{G} \begin{pmatrix} x^{(n-1)} \\ x^{(n)} \end{pmatrix} + \begin{pmatrix} 0 \\ K_{RJ} \end{pmatrix}, \quad (17)$$

$$\text{where } \widehat{G} = \begin{pmatrix} 0 & I \\ F_{RJ} & G_{RJ} \end{pmatrix}$$

The above equation converges to the exact solution if $\sigma(\widehat{G}) < 1$, i.e. the spectra radius of $\widehat{G}$ is less than one. In order to solve the spectra radius of $\widehat{G}$, first we have to solve the eigenvalues λ of $\widehat{G}$.

$$\sigma(\widehat{G}) < 1 \text{ iff all roots } \lambda_{RJ} \text{ of } det({\lambda_{RJ}}^2 I - \lambda_{RJ}G_{RJ} - F_{RJ}) = 0 \quad (18)$$

$$\begin{aligned} &i.e.\ det({\lambda_{RJ}}^2 I - \lambda_{RJ}G_{RJ} - F_{RJ}) = 0 \\ &\Rightarrow det({\lambda_{RJ}}^2 I - \lambda_{RJ}[(1+a-b_1)I + b_1 H_{RJ}] + aI) = 0 \\ &\Rightarrow det({\lambda_{RJ}}^2 I - \lambda_{RJ}(1+a-b_1)I - \lambda_{RJ}b_1 H_{RJ} + aI) = 0 \\ &\Rightarrow det(-\lambda_{RJ}b_1[-\frac{\lambda_{RJ}}{b_1}I + \frac{(1+a-b_1)}{b_1}I + H_{RJ} - \frac{a}{\lambda_{RJ}b_1}I] = 0 \\ &\Rightarrow (-\lambda_{RJ}b_1)^n det(H_{RJ} + \frac{(1+a-b_1)}{b_1}I - \frac{\lambda_{RJ}^2 + a}{\lambda_{RJ}b_1}I) = 0 \end{aligned}$$

$$\Rightarrow det(H_{RJ} + \frac{(1+a-b_1)}{b_1}I - \frac{\lambda_{RJ}^2 + a}{\lambda_{RJ}b_1}I) = 0, since(-\lambda_{RJ}b_1)^n \neq 0 \quad (19)$$

Thus, the eigenvalues λ_{RJ} of $\widehat{G}$ are related to the eigenvalues μ_{RJ} of H_{RJ} with H_J is

$$\mu_{RJ} + \frac{(1+a-b_1)}{b_1} = \frac{(a + {\lambda_{RJ}}^2)}{\lambda_{RJ}b_1} \quad (20)$$

As the image of the circle, Let the eigenvalue $\lambda_{RJ} = ve^{i\theta} = v(\cos\theta + i\sin\theta)$ is the ellipse, then substituting this in equation (18), we obtain

$$\begin{aligned} &\mu_{RJ} + \frac{(1+a-b_1)}{b_1} = \frac{(ve^{i\theta})^2 + a}{ve^{i\theta}b_1}. \\ &\Rightarrow \mu_{RJ} + \frac{1+a-b_1}{b_1} = \frac{ve^{i\theta}}{b_1} + \frac{a}{b_1 ve^{i\theta}} \\ &\Rightarrow \mu_{RJ} + \frac{1+a-b_1}{b_1} = \frac{v(\cos\theta + i\sin\theta)}{b_1} + \frac{a(\cos\theta - i\sin\theta)}{b_1 v} \\ &\Rightarrow \mu_{RJ} = -\frac{(1+a-b_1)}{b_1} + \frac{v\cos\theta}{b_1} + i\frac{v\sin\theta}{b_1} + \frac{a\cos\theta}{b_1 v} - i\frac{a\sin\theta}{vb_1} \end{aligned}$$

$$\therefore \mu_{RJ} = \frac{1}{b_1}(v + \frac{a}{v})\cos\theta - \frac{1+a-b_1}{b_1} + i\frac{1}{b_1}(v - \frac{a}{v})\sin\theta \quad (21)$$

i.e. $Re\mu_{RJ} = \frac{1}{b_1}(v + \frac{a}{v})\cos\theta - \frac{(1+a-b_1)}{b_1}$

$$\Rightarrow \cos\theta = \frac{Re\mu_{RJ} + \frac{(1+a-b_1)}{b_1}}{\frac{1}{b_1}(v + \frac{a}{v})} \quad (22)$$

$Im\mu_{RJ} = \frac{1}{b_1}(v - \frac{a}{v})\sin\theta$

$$\Rightarrow \sin\theta = \frac{IM\mu_{RJ}}{\frac{1}{b_1}(v + \frac{a}{v})} \quad (23)$$

We know that $\cos^2\theta + \sin^2\theta = 1$

$$[\frac{Re\mu_{RJ} + \frac{(1+a-b_1)}{b_1}}{\frac{1}{b_1}(v + \frac{a}{v})}]^2 + [\frac{Im\mu_{RJ}}{\frac{1}{b_1}(v - \frac{a}{v})}]^2 = 1 \quad (24)$$

$$centre = c(h,k) = (-\frac{(1+a-b_1)}{b_1}, 0) \quad (25)$$

$$centre = c(h,k) = (-\frac{(1+a-b_1)}{b_1}, 0) \quad (26)$$

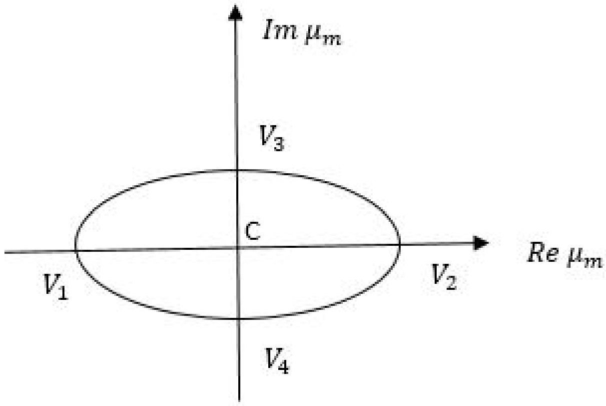
$$\text{Length of semi-major axis } = a^/ = \frac{1}{b_1}(v + \frac{a}{v}) \quad (27)$$

$$\text{Length of semi-minor axis } = b^/ = \frac{1}{b_1}(v - \frac{a}{v}) \quad (28)$$

$$\text{Foci } = F_1 = (h-c, 0) = (-\frac{1+a-b_1}{b_1} - \frac{2\sqrt{a}}{b_1}, 0) = (\alpha, 0) \quad (29)$$

$$\text{Foci } = F_2 = (h+c, 0) = (-\frac{1+-b_1}{b_1} + \frac{2\sqrt{a}}{b_1}, 0) = (\beta, 0) \quad (30)$$

$$\begin{aligned} v_1 &= (h - a', 0) = (-\frac{1+a-b_1}{b_1} - \frac{1}{b_1}(v + \frac{a}{v}), 0) \\ v_2 &= (h + a', 0) = (-\frac{1+a-b_1}{b_1} + \frac{1}{b_1}(v + \frac{a}{v}), 0) \\ v_3 &= (h, k + b') = (-\frac{1+a-b_1}{b_1}, \frac{1}{b_1}(v - \frac{a}{v})) \\ v_4 &= (h, k - b') = (-\frac{1+a-b_1}{b_1}, -\frac{1}{b_1}(v - \frac{a}{v})) \end{aligned}$$



Before we prove the theorem let us prove the following Lemmas.

Lemma 2: If μ_{RJ} is real, then $\alpha \leq \mu_{RJ} \leq \beta < 1$, for any foci α and β which are real.

Proof: We know that μ_{RJ} is a real number. We require that $\alpha \leq \mu_{RJ} \leq \beta < 1$, for any foci α and β which are real. The proof is as follows:

$$\Rightarrow \mu_{RJ} = \frac{1}{b_1}(v+\frac{a}{v})\cos\theta - \frac{1+a-b_1}{b_1},$$

since μ_{RJ} is a real number. In this equation θ varies.

$$-\frac{1}{b_1}(v+\frac{a}{v}) - \frac{1+a-b_1}{b_1} \leq \mu_m \leq \frac{1}{b_1}(v+\frac{a}{v})\cos\theta - \frac{1+a-b_1}{b_1},$$

since $-1 \leq \cos\theta \leq 1$

$$\Rightarrow \alpha \leq \mu_{RJ} \leq \mu_J \leq \beta < 1 \tag{31}$$

Because α and β from equation (26) and (27) and to be convergent $\rho(H_{RJ}) < 1$ so all the eigenvalues must be less than 1. ■

Lemma 3: If the eigenvalues μ_{RJ} of $H_{RJ} < 1$ are real and lie in the interval.

Proof: $\alpha \leq \mu_{RJ} \leq \mu_J \leq \beta < 1$, then the optimal choices of a and b_1 must satisfy the following conditions:

$$a) \quad v^2 = a \tag{32}$$

$$b) \quad \frac{\alpha+\beta}{2} = -\frac{1+a-b_1}{b_1} \tag{33}$$

$$c) \quad \frac{\beta-\alpha}{2} = \frac{2v}{b_1} \tag{34}$$

$$d) \quad 2v = \frac{\beta-\alpha}{2-(\alpha+\beta)}(1+v^2) \tag{35}$$

Given: the eigenvalues μ_{RJ} of $H_{RJ} < 1$ are real and lie in the interval $\alpha \leq \mu_{RJ} \leq \mu_J \leq \beta < 1$. Required: proof of a) until d). Proof :

a) we know μ_m is real, then $\frac{1}{b_1}(v-\frac{a}{v})\sin\theta = 0$, we have $\frac{1}{b_1}\sin\theta = 0$ or $(v-\frac{a}{v}) = 0$, so we get $v^2 = a$ or $\sin\theta = 0$, from the second equation we have $\theta = 2\pi, n = 0,1,2....$ Therefore $V^2 = a$.

b) From the (26) and (27) and from Lemma 2(a) ,we get:

$$\alpha = \frac{-2v}{b_1} - \frac{1+a-b_1}{b_1} \text{ and } \beta = \frac{2v}{b_1} - \frac{1+a-b_1}{b_1}$$

$$\Rightarrow \frac{\alpha+\beta}{2} = -\frac{1+a-b_1}{b_1} \text{ (mid point formula)}$$

c) We know from (b) above we have $\alpha = \frac{-2v}{b_1} - \frac{1+a-b_1}{b_1}$ and $\beta = \frac{2v}{b_1} - \frac{1+a-b_1}{b_1}$, then one can get

$$\frac{\beta-\alpha}{2} = \frac{2v}{b_1}$$

d) From Lemma 2 (b), we have

$$\frac{\alpha+\beta}{2} = -\frac{1+a-b_1}{b_1}$$

$$\Rightarrow 1-\frac{\alpha+\beta}{2} = 1-(-\frac{1+a-b_1}{b_1})$$

$$\Rightarrow \frac{2-(\alpha+\beta)}{2} = 1+(\frac{1+a-b_1}{b_1}) = \frac{1+a}{b_1} \tag{36}$$

Divide equation (31) by (33),we get

$$\Rightarrow \frac{\frac{\beta-\alpha}{2}}{\frac{2-(\alpha+\beta)}{2}} = \frac{\frac{2v}{b_1}}{\frac{1+a}{b_1}}$$

$$\Rightarrow \frac{\beta-\alpha}{2-(\alpha+\beta)} = \frac{2v}{1+a}$$

$$\Rightarrow 2v = \frac{\beta-\alpha}{2-(\alpha+\beta)}(1+v^2)$$

■

Lemma 4: If $\overline{\mu}_{RJ}$ is the spectral radius of H_{RJ}, then

$$\overline{\mu}_{RJ} = \frac{\beta-\alpha}{2-(\alpha+\beta)} \tag{37}$$

Proof: Given: μ_{RJ} is the spectral radius of H_{RJ}. Required: $\mu_{RJ} = \frac{\beta-\alpha}{2-(\alpha+\beta)}$. Proof: we know that $\mu_{RJ} = \frac{1}{b_1}(v+\frac{a}{v})\cos\theta - \frac{1+a-b_1}{b_1}$. By definition of derivative of functions in calculus

$$\frac{d\mu_{RJ}}{d\theta} = \frac{d}{d\theta}[\frac{1}{b_1}(v+\frac{a}{v})\cos\theta - \frac{1+a_1-b_1}{b_1}] = -\frac{1}{b_1}(v+\frac{a}{v})\sin\theta$$

To calculate the maximum and minimum value, the above equation equates to zero.

$$-\frac{1}{b_1}(v+\frac{a}{v})\sin\theta = 0$$

$$\Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi...$$

When $\theta = 0$, then $\mu_{RJ} = \frac{1}{b_1}(v+\frac{a}{v}) - \frac{1+a-b_1}{b_1}$
When $\theta = \pi$, then $\mu_{RJ} = -\frac{1}{b_1}(v+\frac{a}{v}) - \frac{1+a-b_1}{b_1}$
When $\theta = 2\pi$, then $\mu_{RJ} = \frac{1}{b_1}(v+\frac{a}{v}) - \frac{1+a-b_1}{b_1}$
From the above the maximum value occurs at $\theta = 0$ and 2π

$$\Rightarrow \max_{i=1}^{n}(\mu_{RJ}) = \frac{1}{b_1}(v+\frac{a}{v}) - \frac{1+a-b_1}{b_1} = \beta$$

The minimum value occurs at

$$\theta = \pi \Rightarrow \min_{i=1}^{n}\mu_{RJ} = -\frac{1}{b_1}(v+\frac{a}{v}) - \frac{1+a-b_1}{b_1} = \alpha$$

$$\Rightarrow \overline{\mu}_{RJ} = \max_{i=1}^{n} \mid \mu_{RJ} \mid = \max_{i=1}^{n} \mid \frac{1}{b_1}(v+\frac{a}{v})\cos\theta - \frac{1+a-b_1}{b_1} \mid$$

$$= \frac{1}{b_1}(v+\frac{a}{v}) - \frac{1+a-b_1}{b_1}, since -1 \leq \cos\theta \leq 1 \text{ and}$$

$$\Rightarrow -\overline{\mu}_{RJ} = \min_{i=1}^{n}(-\mid \mu_{RJ} \mid)$$
$$= \min - \mid -\frac{1}{b_1}(v+\frac{a}{v})\cos\theta - \frac{1+a-b_1}{b_1} \mid$$
$$= \frac{1}{b_1}(v+\frac{a}{v}) - \frac{1+a-b_1}{b_1}$$
$$\Rightarrow \overline{\mu}_{RJ} = \frac{2v}{b_1} - \frac{1+a-b_1}{b_1} = \frac{2v}{b_1} - \frac{1+a}{b_1} + 1 =$$
$$\frac{\beta-\alpha}{2} - \frac{2-(\alpha+\beta)}{2} + 1 = \beta$$

by equation (31) and (33). Therefore $\overline{\mu}_{RJ} = \beta$.

$$\Rightarrow -\overline{\mu}_{RJ} = \min_{i=1}^{n}(-\mid \mu_{RJ} \mid) = -\frac{1}{b_1}(v+\frac{a_1}{v}) - \frac{1+a_1-b_1}{b_1} =$$
$$-\frac{2v}{b_1} - \frac{1+a_1}{b_1} + 1 = \alpha$$

Therefore $\overline{\mu}_{RJ} = -\alpha$. From the previous two results, we obtain $2\overline{\mu}_{RJ} = \beta - \alpha$ and $2 = 2-(\alpha+\beta)$. Then divide the previous two equations, we get $\overline{\mu}_{RJ} = \frac{\beta-\alpha}{2-(\alpha+\beta)}$.

Now let us determine the values of a and b_1. First, let us find a from Lemma 3d equation (32)

$$\Rightarrow 2v = \frac{\beta-\alpha}{2-(\alpha+\beta)}(1+v^2)$$

by lemma 3d, we have

$$\Rightarrow 2v = \overline{\mu}_{RJ}(1+v^2)$$
$$\Rightarrow \overline{\mu}_{RJ}v^2 - 2v + \overline{\mu}_{RJ} = 0.$$

This is the equation of quadratic whose graph is a parabola and the minimum value occurs at p=$(\frac{1}{\overline{\mu}_{RJ}}, \frac{\overline{\mu}_{RJ}{}^2-1}{\overline{\mu}_{RJ}})$ since $\overline{\mu}_{RJ} > 0$. One can solve by quadratic formula of the above equation:

$$v = \frac{2 \pm \sqrt{4-4\overline{\mu}_{RJ}{}^2}}{2\mu_{RJ}} = \frac{1 \pm \sqrt{1-\overline{\mu}_{RJ}{}^2}}{\overline{\mu}_{RJ}}$$
$$\Rightarrow v_1 = \frac{1+\sqrt{1-\overline{\mu}_{RJ}{}^2}}{\overline{\mu}_{RJ}}$$

and $v_2 = \frac{1-\sqrt{1-\overline{\mu}_{RJ}{}^2}}{\overline{\mu}_{RJ}}$

The smallest value is $\Rightarrow v_2 = \frac{1-\sqrt{1-\overline{\mu}_{RJ}{}^2}}{\overline{\mu}_{RJ}}$. Let $1+v^2 = \omega \Rightarrow a = \omega - 1$

$$\Rightarrow 1+v^2 = \frac{2}{1+\sqrt{1-\overline{\mu}_{RJ}^2}}$$
$$\therefore a = \frac{\overline{\mu}_{RJ}^2}{(1+\sqrt{1-\overline{\mu}_{RJ}^2})^2},$$

since $a = v^2$.

Secondly, let us find $b_1 \Rightarrow b_1 = \frac{4v}{\beta-\alpha}$ by using equation (27)

$$\Rightarrow b_1 = \frac{4v}{\beta-\alpha} = \frac{2\mu_{RJ}(1+v^2)}{\beta-\alpha}$$
$$\therefore b_1 = \frac{4}{(1+\sqrt{1-\mu_{RJ}^2})(2-(\alpha+\beta))}.$$

■

Lemma 5: If matrix A is positive definite matrix and if H_{RJ} is Jacobi iterative matrix, then $\beta = -\alpha = \overline{\mu}_{RJ} = \sigma(H_{RJ})$.

Proof: Given: matrix A is positive definite matrix and if H_{RJ} is Jacobi iterative matrix. Required: $\beta = -\alpha = \overline{\mu}_{RJ} = \sigma H_{RJ}$. Proof: In order to prove this Lemma, we have to use Lemma 3

$$\Rightarrow \overline{\mu}_{RJ} = \max_{i=1}^{n} \mid \mu_{RJ} \mid = \frac{1}{b_1}(v+\frac{a}{v}) - \frac{1+a-b_1}{b_1} = \beta$$
$$\Rightarrow -\overline{\mu}_{RJ} = -\frac{1}{b_1}(v+\frac{a}{v}) - \frac{1+a-b_1}{b_1} = \alpha$$
$$\therefore \beta = -\alpha = \overline{\mu}_{RJ}$$

Now we can find the optimal value of a and b_1

i.e. $a = \frac{\overline{\mu}_{RJ}^2}{(1+\sqrt{1-\overline{\mu}_{RJ}^2})^2}$. $b_1 = \frac{2}{1+\sqrt{1-\overline{\mu}_{RJ}^2}}$., since $\beta = -\alpha \Rightarrow \alpha+\beta = 0$. Now let us find second degree of Refinement Jacobi (SDRJ) method:

$$\Rightarrow \frac{1+a-b_1}{b_1} = \frac{\alpha+\beta}{2}$$
$$\Rightarrow \frac{1+a-b_1}{b_1} = \frac{\alpha-\alpha}{2}$$

since $\beta = -\alpha$

$$\Rightarrow (1+a-b_1) = 0$$

From the second degree

$$\Rightarrow x^{(n+1)} = G_{RJ}x^{(n)} + F_{RJ}x^{(n-1)} + k_{RJ}$$
$$\Rightarrow x^{(n+1)} = [(1-b_1+a)I + b_1H_{RJ}]x^{(n)} + (-aI)x^{(n-1)} + b_1C_{RJ}$$
$$\Rightarrow x^{(n+1)} = b_1H_{RJ}x^{(n)} - ax^{(n-1)} + b_1CRJ_{RJ}$$
$$\therefore x^{(n+1)} = b_1[D^{-1}(L+U)x^{(n)} + k_1] - ax^{(n-1)},$$

where

$$a = \frac{\overline{\mu}^2}{(1+\sqrt{1-\overline{\mu}^2})^2}, \quad b_1 = \frac{2}{1+\sqrt{1-\overline{\mu}^2}}.$$

■

III. Relationship between spectral radius

As we have seen above the spectral radius of

- First degree Jacobi method(FDJ) is $\overline{\mu}$.
- First degree Refinement Jacobi method(FDJ) is $\overline{\mu}_{RJ}$.= $\overline{\mu}$.^{2}
- Second degree Jacobi method(SDJ) is $\sqrt{a} = \frac{\overline{\mu}}{1+\sqrt{1-\overline{\mu}^2}}$.
- Second degree Refinement Jacobi method(SDGJ) is $a = \frac{\overline{\mu}^2}{(1+\sqrt{1-\overline{\mu}^2})^2}$.

That is one can see $\frac{\overline{\mu}^2}{(1+\sqrt{1-\overline{\mu}^2})^2} \leq \frac{\overline{\mu}}{1+\sqrt{1-\overline{\mu}^2}} \leq \overline{\mu}$ since $1+\sqrt{1-\overline{\mu}^2} > 0$ and also $\overline{\mu}_{RJ} \leq \overline{\mu}$ since $0 < \overline{\mu} < 1$.

IV. Numerical Examples

a. Solve the following SDD matrix using FDJ ,FDGJ ,SDJ and SDRJ iterative methods.

$4x_1 - x_2 - x_3 = 3$

$-2x_1 + 6x_2 + x_3 = 9$

$-x_1 + x_2 + 7x_3 = -6$

Solution: all results are based on the given data and we get the spectral radius as follows

Method	FDJ	FDRJ	SDJ	SDRJ
Spectral radius	0.4295	0.1845	0.2257	0.0931

b. Solve the following PD matrix using FDJ, FDGJ, SDJ and SDRJ iterative methods.

$3x_1 - x_2 - x_3 = 1$

$-x_1 + 3x_2 + x_3 = 3$

$2x_1 + x_2 + 4x_3 = 7$

Solution: all results are based on the given data we get the spectral radius as follows

Method	FDJ	FDRJ	SDJ	SDRJ
Spectral radius	0.3333	0.1111	0.1716	0.0557

The detailed experimental results are written in the appendix. Table I shows that FDRJ method converges faster than FDJ method for SDD matrix. Table II shows that SDRJ method converges faster than the SDJ method for SDD matrix. Table III shows that FDRJ method converges faster than FDJ method for PD matrix. Table IV shows that SDRJ method converges faster than the SDJ method for PD matrix.

V. Conclusions

As we have seen in this report for SDD and PD matrix, we can notice that FDJ, FDRJ and SDJ are reasonable to approximate the exact solution of system of linear equations at a certain given condition. But they are relatively slow to converge to the exact solution. However, the Second degree of refinement Jacobi iterative method for solving system of linear equations are reasonable and efficient way of approximating the exact solution of system of linear equations. Numerical results of spectral radius show that, SDRJ methods converge with a small number of iteration steps for solving systems of linear equations.

In general, the results of numerical examples considered clearly demonstrate the accuracy of the methods developed in this paper. It is conjectured that the rate of convergence of the method that developed in this paper can be further enhanced by using extrapolating techniques.

VI. Acknowledgements

I would like to express my sincere appreciation to Professor Vatti, Bassava Kumar, Department of Engineering mathematics, College of Engineering, Andhra University Visakhapatnam 530 003, India. My Student Gashaye Dessalew and Yilkal Abiyu and also my friend Bekalu Tarekegn, Abraham Kassa and Teshager Assefa for their comments and suggestion for preparation of the work.

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Appendix

TABLE I
All numerical result of SDD matrix for FDJ and FDRJ of Example 1

	First degree Jacobi (FDJ)			First degree refinement Jacobi (FDRJ)		
n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
0	0	0	0	0	0	0
1	0.75	1.5	-0.857143	0.910714	1.892857	-0.964286
2	0.9107515	1.892857	-0.964285	0.99171	1.993622	-0.997451
3	0.982143	1.964286	-0.997448	0.999136	1.999726	-0.999683
4	0.991710	1.993622	-0.997448	0.999895	2.000018	-0.999952
5	0.999044	1.996811	-1.000819	0.999986	2.000008	-0.999993
6	0.998998	1.999818	-0.99968	0.999999	2.000002	-1.000000
7	1.000035	1.999613	-1.000116	1.000001	2.000000	-1.000002
8	0.999873	2.000031	-0.999939	1.000000	2.000000	-1.000000
9	1.00023	1.999948	-1.000022			
10	0.999982	2.000011	-0.999989			
11	1.000006	1.999992	-1.000003			
12	0.999998	2.000003	-0.999997			
13	1.000002	1.999999	-1.000000			
14	1.000000	2.000001	-1.000000			
15	1.000000	2.000000	-1.000000			

TABLE II
All numerical result of SDD matrix for SDJ and SDRJ of Example 1

	Second degree Jacobi (SDJ)			Second degree refinement Jacobi (SDRJ)		
n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
0	0	0	0	0	0	0
1	0.75	1.5	-0.857143	0.910714	1.892857	-0.964286
2	0.945649	1.965467	-1.001276	0.996783	2.003819	-1.002551
3	1.000294	2.000589	-1.00842	1.000043	2.000823	-1.000045
4	1.000052	2.002884	-0.999995	0.999995	2.000064	-0.999962
5	1.000739	1.999994	-1.000097	0.999997	2.000006	-0.999995
6	0.999972	2.000162	-0.999889	1.000000	2.000001	-0.999999
7	1.000043	1.999971	-1.000024	1.000000	2.000000	-1.000000
8	0.99987	2.000031	-0.999939			
9	1.000003	1.999995	-1.000003			
10	0.999998	2.000002	-0.999999			
11	1.000001	1.999999	-1.000001			
12	0.999999	2.000001	-1.000000			
13	1.000000	2.000000	-1.000000			

TABLE III
ALL NUMERICAL RESULT OF POSITIVE DEFINITE (PD) MATRIX FOR FDJ AND FDRJ OF EXAMPLE 2

	First degree Jacobi (FDJ)			First degree refinement Jacobi (FDRJ)		
n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
0	0	0	0	0	0	0
1	0.333333	1.000000	1.75	1.250000	0.527778	1.333333
2	1.250000	0.527778	1.333333	0.988426	0.986883	1.030093
3	0.953704	0.972222	0.993056	0.998392	0.998864	1.000643
4	0.988426	0.986883	1.030093	1.000113	0.999583	1.000270
5	1.005659	0.986111	1.009066	0.999986	0.999967	1.000039
6	0.998392	0.998864	1.000643	0.999998	0.999968	1.000038
7	0.999836	0.999250	1.001088	0.999999	0.999998	0.999999
8	1.000113	0.999583	1.000270	1.000000	1.000000	1.000000
9	0.999951	0.999948	1.000048			
10	0.999999	0.999968	1.000038			
11	1.000002	0.999987	1.000009			
12	0.999998	0.999997	1.000005			
13	0.999999	0.999999	1.000003			
14	0.999999	0.999998	1.000002			
15	1.000000	0.999999	1.000001			
16	1.000000	1.000000	1.000000			

TABLE IV
ALL NUMERICAL RESULT OF POSITIVE DEFINITE (PD) MATRIX FOR SDJ AND SDRJ OF EXAMPLE 2

	Second degree Jacobi (SDJ)			Second degree refinement Jacobi (SDFJ)		
n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
0	0	0	0	0	0	0
1	0.333333	1.000000	1.75	1.250000	0.527778	1.333333
2	1.287113	0.543447	1.372920	0.991546	0.989998	1.033342
3	0.991088	0.970548	0.947441	0.996801	1.001843	0.998534
4	0.963327	1.028536	1.001098	1.000214	0.999651	1.000073
5	1.010436	0.987910	1.013096	1.000019	0.999970	1.000030
6	1.001434	0.998240	0.997707	0.999996	1.000003	1.000000
7	0.998299	1.001638	0.999326	1.000000	1.000000	1.000000
8	1.000288	0.999700	1.000522			
9	1.000127	0.999871	0.999949			
10	0.999930	1.000070	0.999952			
11	1.000004	0.999996	1.000020			
12	1.000008	0.999992	1.000000			
13	0.999997	1.000003	0.999997			
14	1.000000	1.000000	1.000001			
15	1.000000	1.000000	1.000000			