Terwilliger Algebras of Group Association Schemes of Matrix Groups

Nur Hamid[1]

***Abstract*—This paper investigates the Terwilliger algebras of some group association schemes related to matrix groups. We obtain the structure of the Terwilliger algebras for the general and the special linear group of 2×2 matrices over the field of order 5. In particular, we determine the Wedderburn decomposition of these algebras.**

I. Introduction

THE Terwilliger algebra, which was known as the subconstituent algebra, was introduced by P. Terwilliger [9]. Terwilliger provided a method for studying association schemes and applied the method to the $P-$ and $Q-$polynomial schemes. The Terwilliger algebra is an important tool for investigating association schemes.

Several previous studies of the Terwilliger algebra of some group association schemes have been done (see [3], [5], [2]). The initial investigation of the Terwilliger algebra of group association schemes can be found in [3]. Balmaceda and Oura [1] continued the investigation of the Terwilliger algebras for the groups S_5 and A_5. These groups are the first nontrivial case for the family of symmetric and alternating groups. The investigation was then conducted for the group S_6, A_6, and $PSL(2,7)$ in [5]. The Terwilliger algebra was investigated over a positive characteristic field in [6].

We can find the exploration of the structure of the Terwilliger algebras over several different types of finite groups of order at most 64 in [4]. More information can also be seen in [2]. This study aimed to determine the structure of the Terwilliger algebra from the finite group of association schemes of matrix groups. Readers may refer [7], [10] for the current research. We use SageMath [8] for the computation.

Although there have been results of Terwilliger algebra of group association schemes, the investigation of Terwilliger algebra for matrix groups is not much. This encourages us to observe the Terwilliger algebra of group association schemes for other matrix groups with bigger orders than in [2].

The groups investigated in this study are the general linear group $GL(2,5)$ of all 2×2 matrices over the field of order 5 whose determinants are not equal to zero and its subgroup $SL(2,5)$ of matrices whose determinants are 1. The groups $GL(2,5)$ and $SL(2,5)$ are of degrees 480 and 120, respectively. The numbers of their conjugacy classes are 24 and 9.

We begin with the definition of a group association scheme.

Nur Hamid is with Mathematics Education, Nurul Jadid University, Indonesia. email: nurhamid@unuja.ac.id

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Definition 1.1: Let G be a finite group and $C_0, C_1, \dots, C_d$ be the conjugacy classes of G. Define the relation R_i on G by

$$(x,y) \in R \Leftrightarrow yx^{-1} \in C_i.$$

Then $\mathcal{X}(G) = (G, \{R_i\})$ becomes a commutative association scheme, called the group association scheme of G.

We correspond each R_i to an adjacency matrix A_i of size $|G| \times |G|$ defined as:

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}.$$

The adjacency matrices $A_0, A_1, \dots, A_d$ form a basis for an algebra $\mathcal{A}$ over $\mathbb{C}$, called the Bose-Mesner algebra, and satisfy

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k.$$

The numbers p_{ij}^k are called the intersection numbers of the group association scheme $\mathcal{X}(G)$ and given by

$$p_{ij}^k = |\{(x,y) \in C_i \times C_j | xy = z \text{ for a fixed } z \in C_k\}|.$$

The algebra $\mathcal{A}$ has a second basis from its primitive idempotents $E_0, E_1, \dots, E_d$ which satisfy

$$E_i \circ E_j = \frac{1}{|G|} \sum_{k=0}^{d} q_{ij}^k E_k$$

where $\circ$ denotes the entry-wise multiplication and q_{ij}^k are the nonnegative real numbers, called the Krein parameters.

Let E_i^* and A_i^* with $i = 0, 1, \dots, d$ be the $|G| \times |G|$ diagonal matrices defined by:

$$(E_i^*)_{x,x} = \begin{cases} 1 & \text{if} \quad x \in C_i, \\ 0 & \text{if} \quad x \notin C_i, \end{cases}$$

$$(A_i^*)_{x,x} = |G|(E_i)_{e,x}$$

where e is the identity of G and $x \in G$. Then $\mathcal{A}^* = \langle E_0^*, E_1^*, \dots, E_d^* \rangle = \langle A_0^*, A_1^*, \dots, A_d^* \rangle$ is an algebra over $\mathbb{C}$ which is called the dual Bose-Mesner algebra of $\mathcal{X}(G)$.

Let $\mathcal{M}_i$ be the full matrix algebra over $\mathbb{C}$ of degree i. As the main result of this paper, we determine the Wedderburn decomposition

$$T(GL(2,5)) \cong \mathcal{M}_4 \oplus \mathcal{M}_6 \oplus \mathcal{M}_{10} \oplus \mathcal{M}_{16} \oplus \mathcal{M}_{16} \oplus \mathcal{M}_{20} \oplus \mathcal{M}_{24} \oplus \mathcal{M}_{24}$$

and

$$T(SL(2,5)) \cong M_1 \oplus \mathcal{M} \oplus \mathcal{M}_3 \oplus \mathcal{M}_7 \oplus \mathcal{M}_7 \oplus \mathcal{M}_8 \oplus \mathcal{M}_9.$$

G	$\|G\|$	$\dim T_0(G)$	$\dim T(G)$	$\dim \tilde{T}(G)$
$GL(2,5)$	480	2136	2216	2336
$SL(2,5)$	120	261	262	296

TABLE I
DIMENSIONS OF $T_0(G), T(G)$, AND $\tilde{T}(G)$

We summarize the dimensions obtained in this paper in Table I.

We present the representatives and the sizes of the conjugacy classes of $GL(2,5)$ and $SL(2,5)$. The ordering of the conjugacy classes is inferred from SageMath [8].

1) $GL(2,5)$

rep. C_i	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$
$\|C_i\|$	1	24	1
rep. C_i	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix}$
$\|C_i\|$	24	1	24
rep. C_i	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$
$\|C_i\|$	1	24	20
rep. C_i	$\begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$
$\|C_i\|$	20	10	20
rep. C_i	$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}$
$\|C_i\|$	20	20	20
rep. C_i	$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$
$\|C_i\|$	20	20	20
rep. C_i	$\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$
$\|C_i\|$	30	30	
rep. C_i	$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$
$\|C_i\|$	30	30	30

2) $SL(2,5)$

rep. C_i	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 \\ 3 & 3 \end{pmatrix}$
$\|C_i\|$	1	12	12
rep. C_i	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$
$\|C_i\|$	1	12	12
rep. C_i	$\begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$
$\|C_i\|$	20	20	30

We close this section by defining the Terwilliger algebra.

Definition 1.2: Let G be a finite group. The Terwilliger algebra, denoted by $T(G)$, of a group association scheme $\mathcal{X}(G)$ is an algebra over $\mathbb{C}$ generated by $\mathcal{A}$ and $\mathcal{A}^*$.

II. BOUNDS ON T

First, we start with the definitions of the space of triple product and the centralizer algebra.

Definition 2.1: Let G be a finite group and $T(G)$ be the Terwilliger algebra for the group G. The space $T_0(G)$ is a subspace of $T(G)$ spanned by the triple matrix products $E_i^* A_j E_k^*$ for $0 \leq i,j,k \leq d$ over $\mathbb{C}$.

Definition 2.2: Let G be a finite group that acts on itself by conjugation. The centralizer algebra $\tilde{T}(G)$ is the set of all $|G| \times |G|$ matrices over $\mathbb{C}$ that commute with all π_g where $(\pi)_{xy} = 1$ if $gxg^{-1} = y$ and 0, otherwise.

In [3], the bounds on the dimension of the Terwilliger algebra T were given as:

$$\dim T_0(G) \leq \dim T(G) \leq \dim \tilde{T}(G). \tag{1}$$

The dimension of $T_0(G)$ is given by the number of nonzero matrices products $E_i^* A_j E_k^*$. In the other words, we can say

$$\dim T_0(G) = |\{(i,j,k) \mid E_i^* A_j E_k^* \neq \mathbf{0}\}|. \tag{2}$$

The dimension formula for $\tilde{T}(G)$ is

$$\dim \tilde{T}(G) = \frac{1}{|G|} \sum_{x \in G} |C_G(x)|^2 = \sum_{i=0}^{d} \frac{|G|}{|C_i|}. \tag{3}$$

We call the group G triply transitive if $\dim T_0(G) = \dim T(G) = \dim \tilde{T}(G)$. From Equations (2) and (3), we obtain the following result.

Proposition 2.3: We have that

1) $\dim T_0(GL(2,5)) = 2136$, $\dim T_0(SL(2,5)) = 261$,
2) $\dim \tilde{T}(GL(2,5)) = 2336$, $\dim \tilde{T}(SL(2,5)) = 296$.

As in [3], the degrees d_i of Wedderburn components of $\tilde{T}(G)$ can be obtained from finding the nonzero row sums of the character table of G. They can be written as

$$d_i = \sum_{j=0}^{d} \overline{\chi_i(u_j)}, \tag{4}$$

where $\chi_i(u_j)$ is the character value at $u_j \in C_j$. Using Equation (4), we have the following proposition.

Proposition 2.4: From (4), we have that

$$\begin{aligned} \tilde{T}(GL(2,5)) \cong & \mathcal{M}_4 \oplus \mathcal{M}_{16} \oplus \mathcal{M}_{16} \oplus \mathcal{M}_{16} \oplus \mathcal{M}_{20} \\ & \oplus \mathcal{M}_{24} \oplus \mathcal{M}_{24} \\ \tilde{T}(SL(2,5)) \cong & \mathcal{M}_6 \oplus \mathcal{M}_7 \oplus \mathcal{M}_7 \oplus \mathcal{M}_9 \oplus \mathcal{M}_9 \end{aligned}$$

III. THE TERWILLIGER ALGEBRAS

In this section, we present the main results of this study. We begin with the dimension of $T(G)$.

Theorem 3.1: The dimensions of $T(GL(2,5))$ and $T(SL(2,5))$ are given as follows.

1) $\dim T(GL(2,5)) = 2216$
2) $\dim T(SL(2,5)) = 262$

Proof: The proof is done by direct calculations with the same method as written in [2], [5]. We obtain the linearly independent elements of the set containing $E_i^* A_j E_k^*$ and $E_i^* A_j E_k^* \cdot E_k^* A_l E_m^* = E_i^* A_j E_k^* A_l E_m^*$. By direct calculation, the product of more than two matrices of the form $E_i^* A_j E_k^*$ does not afford a new linear independent element. This set of linearly independent elements provides a basis for T. ∎

We show the distribution of basis elements for $T_0(G), T(G)$, and $\tilde{T}(G)$ by the square $d+1$ distribution matrices indexed by the conjugacy classes of G. Since the matrices are symmetric, we omit the entry below diagonal for simplicity. Note that neither $GL(2,5)$ nor $SL(2,5)$ is triply transitive. For convenience, we divide the distribution matrix for $GL(2,5)$ by 3 block matrices.

1) $GL(2,5)$

(C_i, C_j)-position for $1 \leq i, j \leq 12$.

$$\begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&1&1 \\ &6,2,0&1&6,2,0&1&6,2,0&1&6,2,0&4&4&4&4 \\ &&1&1&1&1&1&1&1&1&1&1 \\ &&&6,2,0&1&6,2,0&1&6,2,0&4&4&4&4 \\ &&&&1&1&1&1&1&1&1&1 \\ &&&&&6,2,0&1&6,2,0&4&4&4&4 \\ &&&&&&1&1&1&1&1&1 \\ &&&&&&&6,2,0&4&4&4&4 \\ &&&&&&&&5&5&5&5 \\ &&&&&&&&&5&5&5 \\ &&&&&&&&&&5&5 \\ &&&&&&&&&&&5 \end{pmatrix}$$

(C_i, C_j)-position for $1 \leq i \leq 12$ and $13 \leq j \leq 24$.

$$\begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&1&1 \\ 4&4&4&4&4&4&5,0,1&5,0,1&5,0,1&5,0,1&5,0,1&5,0,1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 4&4&4&4&4&4&5,0,1&5,0,1&5,0,1&5,0,1&5,0,1&5,0,1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 4&4&4&4&4&4&5,0,1&5,0,1&5,0,1&5,0,1&5,0,1&5,0,1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 4&4&4&4&4&4&5,0,1&5,0,1&5,0,1&5,0,1&5,0,1&5,0,1 \\ 5&5&5&5&5&5&5&5&5&5&5&5 \\ 5&5&5&5&5&5&5&5&5&5&5&5 \\ 5&5&5&5&5&5&5&5&5&5&5&5 \\ 5&5&5&5&5&5&5&5&5&5&5&5 \end{pmatrix}$$

(C_i, C_j)-position for $13 \leq i, j \leq 24$.

$$\begin{pmatrix} 5&5&5&5&5&5&5&5&5&5&5&5 \\ &5&5&5&5&5&5&5&5&5&5&5 \\ &&5&5&5&5&5&5&5&5&5&5 \\ &&&5&5&5&5&5&5&5&5&5 \\ &&&&5&5&5&5&5&5&5&5 \\ &&&&&5&5&5&5&5&5&5 \\ &&&&&&6,1,2&5,2,2&6,1,2&6,1,2&5,2,2&6,1,2 \\ &&&&&&&7,0,2&5,2,2&5,2,2&7,0,2&5,2,2 \\ &&&&&&&&6,1,2&6,1,2&5,2,2&6,1,2 \\ &&&&&&&&&6,1,2&5,2,2&6,1,2 \\ &&&&&&&&&&7,2&5,2,2 \\ &&&&&&&&&&&6,1,2 \end{pmatrix}$$

2) $SL(2,5)$

$$\begin{pmatrix} 1&1&1&1&1&1&1&1&1 \\ &4&4&1&4&4&4&4&5,0,1 \\ &&4&1&4&4&4&4&5,0,1 \\ &&&1&1&1&1&1&1 \\ &&&&4&4&4&4&5,0,1 \\ &&&&&4&4&4&5,0,1 \\ &&&&&&6,0,2&6,0,2&7,0,3 \\ &&&&&&&6,0,2&7,0,3 \\ &&&&&&&&9,1,6 \end{pmatrix}$$

We describe the matrices as follows. The entry with one number means that the dimensions of $T_0(G), T(G)$, and $\tilde{T}(G)$ are the same. The entry, for example $9,1,6$, means that $\dim T_0 = 9$, $\dim T(G) = 9+1$, and $\dim \tilde{T}(G) = 9+1+6$.

From the distribution matrices, we figure out the connection between the dimension produced in the row C_i and C_j with $|C_i| = |C_j|$. The basis elements obtained are always the same. Although we cannot prove this connection for general cases yet, the relation will be very helpful in reducing the computation time if it holds.

We continue the investigation by showing the basis for the center of T. Let $Z(T)$ be the center of T. The center $Z(T)$ consists of block matrices since it contains the diagonal matrices E_i^*. Thus, we have that

$$Z(T) \subseteq \bigoplus_{i=0}^{d} Z(E_i^* T E_i^*).$$

Let $s = \dim Z(T)$. Based on the fact

$$T = \bigoplus_{i=1}^{s} T\varepsilon_i \cong \bigoplus_{i=1}^{s} \mathcal{M}_{d_i}$$

where ε_i denote the primitive central idempotents for T ($\varepsilon_i^2 = \varepsilon_i \neq \mathbf{0}$,$\varepsilon_i\varepsilon_j = \delta_{ij}\varepsilon_i$,$\sum_{i=1}^{s} \varepsilon_i = 1_T$, and $\varepsilon_i \in Z(T)$), we then obtain the structure of T. The following lemma shows the dimension of $Z(T)$ for each group.

Lemma 3.2: For $G = GL(2,5), SL(2,5)$, the dimensions of $Z(T(G))$ are

1) $\dim Z(T(GL(2,5)) = 8$
2) $\dim Z(T(SL(2,5)) = 7$

Proof: The basis of each $T(G)$ is obtained by solving the linear equations system $\{yx_i = x_i y\}$ ranging over all x_i in a basis for $T(G)$ where $y = \sum c_j b_j$ with basis elements b_j and scalars c_j.

■

Let $e_1, \ldots, e_s$ be a basis for $Z(T(G))$. We have that

$$e_i e_j = \sum_{k=1}^{s} t_{ij}^k e_{ij}^k.$$

Set the matrices B_i with the entries $(B_i)_{jk} = t_{ij}^k$. These matrices are simultaneously diagonalizable since they mutually commute. Thus, we can find a matrix P such that

$$P^{-1} B_i P = \begin{pmatrix} v_1(i) & & \\ & \ddots & \\ & & v_s(i) \end{pmatrix}$$

and define the matrix M by $M_{ij} = v_i(j)$. Then, the primitive central idempotents $\varepsilon_1, \ldots, \varepsilon_s$ for $T(G)$ can be obtained by

$$(\varepsilon_1, \ldots, \varepsilon_s) = (e_1, \ldots, e_s) M^{-1}.$$

Theorem 3.3: The degrees of the irreducible complex representations afforded by every primitive central idempotent for each group are as follows.

1) $T(GL(2,5))$

ε_i	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8
deg ε_i	4	6	10	16	16	20	24	24

2) $T(SL(2,5))$

ε_i	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
deg ε_i	1	3	3	7	7	8	9

Proof: We use the fact $T\varepsilon_i \equiv \mathcal{M}_{d_i}(\mathbb{C})$. Then, d_i^2 equals the number of linearly independent elements of the set $\{x_j\varepsilon_i\}$ where x_j are the basis elements of T. ■

We have our main theorem from Theorem 3.3.

Theorem 3.4: We have the following structures

$$\begin{aligned} T(GL(2,5)) \cong & \mathcal{M}_4 \oplus \mathcal{M}_6 \oplus \mathcal{M}_{10} \oplus \mathcal{M}_{16} \oplus \mathcal{M}_{16} \oplus \mathcal{M}_{20} \\ & \oplus \mathcal{M}_{24} \oplus \mathcal{M}_{24} \\ T(SL(2,5)) \cong & \mathcal{M}_1 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_7 \oplus \mathcal{M}_7 \oplus \mathcal{M}_8 \\ & \oplus \mathcal{M}_9 \end{aligned}$$

where M_i is the full matrix algebra over $\mathbb{C}$ of degree i.

We figure out some information from the Wedderburn decomposition of $T(G)$ and $\tilde{T}(G)$. A component $\mathcal{M}_{16}$ of $\tilde{T}(GL(2,5))$ decomposes into two components $\mathcal{M}_6$ and $\mathcal{M}_{10}$ in $T(GL(2,5))$. The sum $\mathcal{M}_6 \oplus \mathcal{M}_9$ in $\tilde{T}(GL(2,5))$ decomposes into $\mathcal{M}_1 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_8$. Combining Theorems 3.1 and 3.3, we have

$$\dim T(G) = \sum_i d_i^2.$$

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