On Reverse Super Edge Bimagic Labeling of Gear Graph, Hibiscus Graph and Dove Tail Graph

Titin Sri Martini, Diari Indriati, Tri Atmojo Kusmayadi

***Abstract*—A simple graph $G(V,E)$ admits a reverse super edge bimagic labeling if there exists a bijection function ζ that taking the vertices $u,w \in V(G)$ that incident on edge uw such that the sum of vertex labels incident on edge uw is subtracted from edge label are two different constants τ_1 and τ_2. The reverse edge bimagic labeling is said to be reverse super edge bimagic labeling if the labels of vertices are $1,2,\ldots,|V(G)|$ and the labels of edges are $|V(G)|+1,|V(G)|+2,\ldots,|V(G)|+|E(G)|$. In this paper we investigate that gear graph, dove tail graph, and hibiscus graph admits a reverse super edge bimagic labeling.**

***Index Terms*—Reverse Super Edge Bimagic Labeling, Gear Graph, Dove Tail, Hibiscus Graph.**

I. Introduction

GRAPH in this paper is undirected, finite, and simple. Labeling of a graph is a function that carries vertices, edges, or both subject to positive or nonnegative integers with certain conditions [1]. The topic of labeling can be found in Gallian [2]. According to [3], if the domain is only a vertex set, then we called vertex labeling. If the domain is an edge set, then we said edge labeling, and if the domain are both vertex set and edge set, then it is namely total labeling. Sedláčk introduced magic labeling [4], then Kotzig and Rosa developed edge magic labeling [5]. In [6], there was a reverse edge magic labeling on lobster graph, path P_2 Cartesian product with cycle with pendant vertex, cycle C_n corona path P_2, ladder corona complete graph K_1, cycle graph C_n with a chord of distance 3, disconnected m cycle C_n were proved. According to Frucht & Harary [7], the corona product of two graphs G_1 and G_2, denoted $G_1 \odot G_2$, is a graph obtained by taking one copy of G_1 and $|V(G_1)|$ copy of G_2 and then joining the ith vertex of G_1 with an edge to every vertex in the ith copy of G_2.

The new idea of labeling then proposed in 2004, it was called edge bimagic labeling. For any $u,w \in V$, and $uw \in E$, edge bimagic labeling is a one-to-one mapping φ that carries vertices and edges onto the integer $\{1,2,\ldots,|V(G)|+|E(G)|\}$ with property $\varphi(u)+\varphi(uw)+\varphi(w)$ are integer constants θ_1 and θ_2 [8]. Babuje, [8], [9], [10], [11], [12] proved bimagic labeling in path graph, disconnected graph, $(3,n)$-kite graph $(n \geq 2)$, the $P_3 \odot K_{1,n}$ (n is even), P_n+N_2 ($n \geq 3, n$ is odd), $P_2 \cup mK_1 + N_2$ $(m \geq 1)$, and P_2+mK_1 $(m \geq 2)$. Amuthavalli then developed edge bimagic labeling into a reverse edge bimagic labeling [13]. According to [13], reverse edge bimagic labeling of graph G is a bijection $\varphi : V(G) \cup E(G) \to \{1,2,3,\ldots,|V(G)|+|E(G)|\}$ such that for each edge $uw \in E(G)$ the value of $\varphi(uw)-(\varphi(u)+\varphi(w))$ is any distinct constants τ_1 and τ_2. In [13], Amuthavalli proved that shadow graph, splitting graph, jellyfish graph, jewel graph are reverse super edge bimagic labeling. Furthermore, in 2022, Martini proved reverse edge bimagic labeling on broom graph and generalized butterfly graph [14]. In this research we found gear graph, dove tail graph, hibiscus graph admits a reverse super edge bimagic labeling.

T. S. Martini, D. Indriati, and T. A. Kusmayadi are with the Universitas Sebelas Maret, Ir. Sutami Street 36A, Surakarta, 57126 Indonesia e-mail: titinsmartini@gmail.com, diari_indri@staff.uns.ac.id, tri.atmojo.kusmayadi@staff.uns.ac.id

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II. Main Result

A. Reverse Edge Bimagic Labeling

Based on [13], reverse edge bimagic labeling of graph G is a bijection $\varphi : V(G) \cup E(G) \to \{1,2,3,\ldots,|V(G)|+|E(G)|\}$ such that for each edge $uw \in E(G)$ the value of $\varphi(uw) - (\varphi(u)+\varphi(w))$ is any distinct constants τ_1 and τ_2. Reverse edge bimagic labeling is called super if the vertices are get labeled the smallest possible integers i.e. $1,2,3,\ldots,|V(G)|$.

B. Reverse Super Edge Bimagic Labeling of Gear Graph

Gear graph G_n is a graph constructed from a wheel graph with inserted a vertex between adjacent vertices in cycle. Gear graph has vertex set $V(G_n)=\{w,v_1,v_2,\ldots,v_{2n-1},v_{2n}\}$, $|V(G_n)|=2n+1$ and edge set $E(G_n)=\{wv_i \mid i \text{ is odd}, i \in [1,2n]\} \cup \{v_iv_{i+1} \mid i \in [1,2n-1]\}$, $|E(G_n)|=3n$ [14]. Gear graph G_n is shown in Fig. 1.

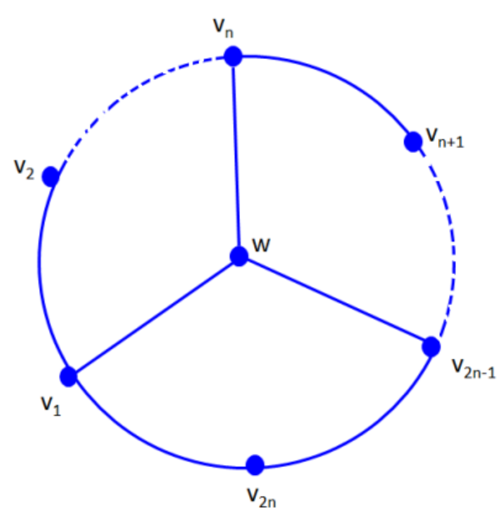

Fig. 1: Gear Graph G_n

Theorem 1. *The Gear graph G_n is reverse super edge bimagic labeling for $n \geq 3$.*

Proof. Let G be a gear graph. We have $V = \{w\} \cup \{v_i \mid i \in [1,2n]\}$ be the vertex set and $E = \{wv_i \mid i \text{ is odd}, i \in [1,2n]\} \cup \{v_iv_{i+1} \mid i \in [1,2n-1]\} \cup \{v_1v_{2n}\}$ be the edge set of G. Then G_n has $2n+1$ vertices and $3n$ edges. Now, we define φ labeling on G_n.

The vertex labels are defined by

$$\varphi(w) = 1,$$
$$\varphi(v_i) = \begin{cases} \lceil \frac{i}{2} \rceil + 1 & 1 \leq i \leq 2n, i \text{ is odd}, \\ n+1+\frac{i}{2} & 1 \leq i \leq 2n, i \text{ is even}. \end{cases}$$

And the edges label are

$$\varphi(wv_i) = 2n+1+\left\lceil \frac{i}{2} \right\rceil, i \text{ is odd},$$
$$\varphi(v_iv_{i+1}) = 3n+2+i, 1 \leq i \leq 2n-1,$$
$$\varphi(v_1v_{2n}) = 3n+2.$$

Then, the constants τ_1 and τ_2 of reverse super edge bimagic are obtain as follows. Consider the edges wv_i, i is odd:

$$\begin{aligned}\varphi(wv_i) - (\varphi(w)+\varphi(v_i)) &= 2n+1+\left\lceil \frac{i}{2} \right\rceil - \left(1+\left\lceil \frac{i}{2} \right\rceil+1\right) \\ &= 2n-1 \\ &= \tau_1.\end{aligned}$$

For edges v_iv_{i+1}, $1 \leq i \leq 2n-1$:
If i is odd,

$$\begin{aligned}&\varphi(v_iv_{i+1}) - (\varphi(v_i)+\varphi(v_{i+1})) \\ &= 3n+2+i-\left(\left\lceil \frac{i}{2} \right\rceil+1+n+1+\frac{i}{2}\right) \\ &= 2n-1 \\ &= \tau_1.\end{aligned}$$

If i is even,

$$\begin{aligned}&\varphi(v_iv_{i+1}) - (\varphi(v_i)+\varphi(v_{i+1})) \\ &= 3n+2+i-\left(n+1+\frac{i}{2}+\left\lceil \frac{i}{2} \right\rceil+1\right) \\ &= 2n-1 \\ &= \tau_1.\end{aligned}$$

Consider the edge v_1v_{2n}:

$$\begin{aligned}&\varphi(v_1v_{2n}) - (\varphi(v_1)+\varphi(v_{2n})) \\ &= 3n+2-\left(\left\lceil \frac{i}{2} \right\rceil+1+n+1+\frac{i}{2}\right) \\ &= n-1 \\ &= \tau_2.\end{aligned}$$

Hence for each edge $wv \in E(G_n)$, $\varphi(wv)-(\varphi(w)+\varphi(v))$ we get two different magic constants $\tau_1 = 2n-1$, $\tau_2 = n-1$ and the vertices of gear graph gets labels $1,2,\ldots,2n+1$. Therefore, the gear graph admits reverse super edge bimagic labeling. Since the gear graph admits reverse super edge bimagic labeling, we call it gear graph is a reverse super edge bimagic graph. Fig. 2 illustrates a reverse super edge bimagic labeling of gear graph G_5. ■

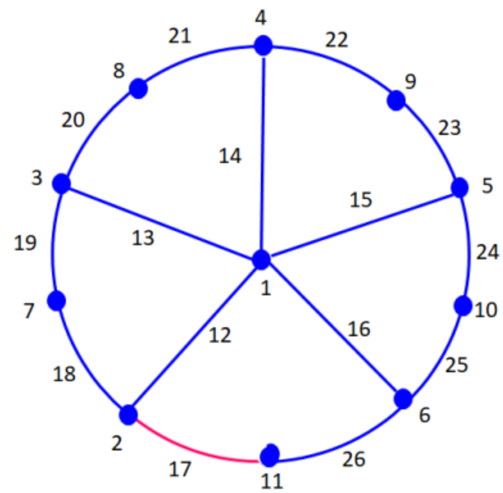

Fig. 2: Reverse super edge bimagic labeling of gear graph G_5 with $\tau_1 = 9, \tau_2 = 4$

C. Reverse Super Edge Bimagic Labeling of Dove Tail Graph

According to [15], dove tail graph is the graph $P_n + K_1, n \geq 2$. The dove tail graph have vertex set $V = \{u, w_1, w_2, \ldots, w_n\}$ and edge set $E = \{uw_i \mid i \in [1,n]\} \cup \{w_iw_{i+1} \mid i \in [1,n-1]\}$. Dove tail graph has $n+1$ vertices and $2n-1$ edges. It is denoted by D_n. The dove tail graph is shown in Fig. 3.

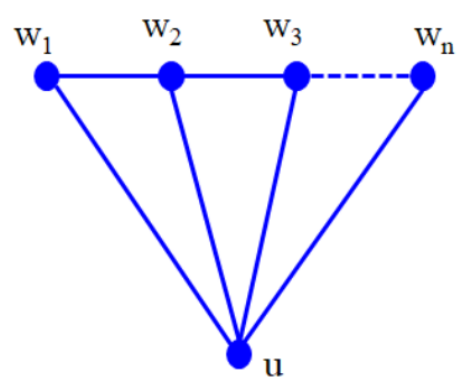

Fig. 3: Dove tail graph D_n

Theorem 2. *The dove tail graph D_n is reverse super edge bimagic labeling for $n \geq 3$.*

Proof. Let H be a dove tail graph. We have $V = \{u\} \cup \{w_i \mid i \in [1,n]\}$ be the vertex set and $E = \{uw_i \mid i \in [1,n]\} \cup \{w_iw_{i+1} \mid \in \in [1,n-1]\}$ be the edge set of H. Then D_n has $n+1$ vertices and $2n-1$ edges. To label vertices and edges, we define bijection function $\zeta : V \cup E \to \{1,2,\ldots,|V|+|E|\}$, then we label as follows. Label of vertices

$$\zeta(u) = 1,$$

$$\zeta(w_i) = i+1.$$

Label of edges

$$\zeta(uw_i) = n+i+1, 1 \leq i \leq n$$

$$\zeta(w_iw_{i+1}) = \begin{cases} 2(n+i), & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor,\ n \text{ is odd,} \\ n+2i+2, & \lceil \frac{n}{2} \rceil \leq i \leq n-1, \quad n \text{ is odd and} \\ & \frac{n}{2} \leq i \leq n-1, \quad n \text{ is even,} \\ 2(n+i)+1, & 1 \leq i \leq \frac{n}{2}-1,\ n \text{ is even.} \end{cases}$$

Then the constants τ_1 and τ_2 obtained as follows.

Consider edges $uw_i, i \leq i \leq n$

$$\begin{aligned} \zeta(uw_i) - (\zeta(u) + \zeta(w_i)) &= n+i+1-(1+i+1) \\ &= n-1 \\ &= \tau_1. \end{aligned}$$

For n is even, consider edges w_iw_{i+1}, $1 \leq i \leq \frac{n}{2}-1$

$$\begin{aligned} &\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) \\ &= 2(n+i)+1-(i+1+(i+1)+1) \\ &= 2n-2 \\ &= \tau_2, \end{aligned}$$

$\frac{n}{2} \leq i \leq n-1$,

$$\begin{aligned} &\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) \\ &= n+2i+2-(i+1+(i+1)+1) \\ &= n-1 \\ &= \tau_1. \end{aligned}$$

For n is odd, consider edges w_iw_{i+1}, $1 \leq i \leq \lfloor \frac{n}{2} \rceil$

$$\begin{aligned} &\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) \\ &= 2(n+i)-(i+1+(i+1)+1) \\ &= 2n-3 \\ &= \tau_2, \end{aligned}$$

$\lceil \frac{n}{2} \rfloor \leq i \leq n-1$,

$$\begin{aligned} &\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) \\ &= n+2i+2-(i+1+(i+1)+1) \\ &= n-1 \\ &= \tau_1. \end{aligned}$$

Hence for each edge $uw \in E(D_n)$, $\zeta(uw) - (\zeta(u) + \zeta(w))$ we get any one of magic constants $\tau_1 = n-1, \tau_2 = 2n-2$ for even n and $\tau_1 = n-1, \tau_2 = 2n-3$ for odd n, and the vertices og dove tail graph gets labels $1, 2, \ldots, n+1$. Therefore the dove tail graph is reverse super edge bimagic labeling, so that the dove tail graph is reverse super edge bimagic graph. ■

Here, in Fig. 4 and Fig. 5 we showed reverse super edge bimagic labelling of dove tail graph D_4 and D_5 respectively.

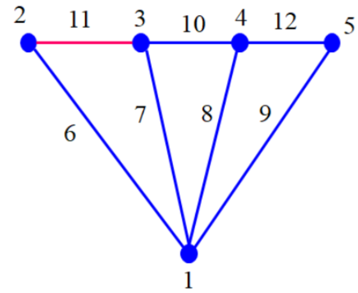

Fig. 4: Reverse super edge bimagic labeling of dove tail graph D_4 with $\tau_1 = 3$ and $\tau_2 = 6$

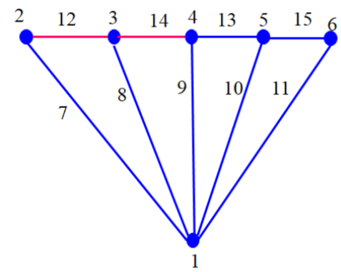

Fig. 5: Reverse super edge bimagic labeling of dove tail graph D_5 with $\tau_1 = 4$ and $\tau_2 = 7$

D. Reverse Super Edge Bimagic Labeling of Hibiscus Graph

The third graph that we investigate is hibiscus graph. According to [16], hibiscus graph $H_f^{(k)}, k \geq 2$ is a graph obtain from $k-cycle$ C_n and adding a vertex in every cycle that connect to the center vertex. The hibiscus graph has vertex set $V = \{v_0\} \cup \{v_i \mid 1 \leq i \leq k\} \cup \{v_i^j \mid 1 \leq i \leq k, j = 1,2,3\}$ and edge set $E = \{v_0v_i \mid i \in [1,k]\} \cup \{v_0v_i^j \mid i \in [1,k], j = 1,3\} \cup \{v_i^jv_i^{j+1} \mid i \in [1,k], j = 1,2\}$. Hibiscus graph has $4k+1$ vertices and $5k$ edges. The hibiscus graph is given in Fig. 6.

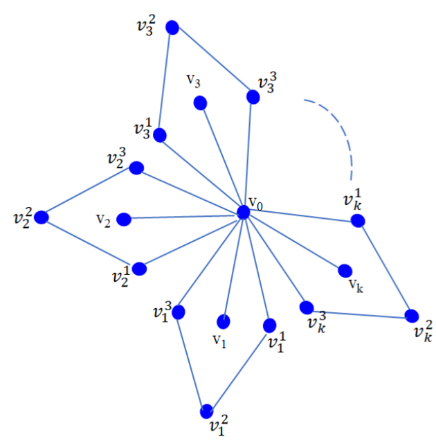

Fig. 6: Hibiscus graph $H_f^{(k)}$

Theorem 3. *The hibiscus graph $H_f^{(k)}$ is reverse super edge bimagic labeling for $k \geq 3$.*

Proof. Let H be a hibiscus graph. We have $V = \{v_0\} \cup \{v_i \mid 1 \leq i \leq k\} \cup \{v_i^j \mid 1 \leq i \leq k, j = 1,2,3\}$ and edge set $E = \{v_0v_i \mid i \in$

$[1,k]\} \cup \{v_0 v_i^j \mid i \in [1,k], j = 1,3\} \cup \{v_i^j v_i^{j+1} \mid i \in [1,k], j = 1,2\}$ be the vertex set of H. The vertices and edges are labeled using bijection function $\gamma : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$, then we label as follows.

Label of vertices are

$$\gamma(v_0) = 1,$$
$$\gamma(v_i) = 3i,$$
$$\gamma(v_i^j) = \begin{cases} 3i + j - 2, & 1 \le i \le k, \quad j = 1,3, \\ 3k + i + 1, & 1 \le i \le k, j = 2. \end{cases}$$

Then label of edges are

$$\gamma(v_0 v_i) = 4k + 3i, \qquad 1 \le i \le k,$$
$$\gamma(v_0 v_i^j) = 4k + 3i + j - 2, \qquad j = 1,3.$$

The label of edges $v_i^j v_i^{j+1}$ are

k	i	j	$v_i^j v_i^{j+1}$
even	$1 \le i \le \frac{k}{2}$	1,2	
odd	$1 \le i \le \lfloor \frac{k}{2} \rfloor$	1,2	$7k + 4i + 2j - 3$
	$i = \lfloor \frac{k}{2} \rfloor + 1$	1	
even	$\frac{k}{2} + 1 \le i \le k$	1,2	$7k + 2j + 4\left(i - \frac{k+2}{2}\right)$
odd	$\lceil \frac{k}{2} \rceil + 1 \le i \le k$	1,2	$5k + 4i + 2j - 4$
odd	$i = \lceil \frac{k}{2} \rceil$	2	

To find constants τ_1 and τ_2 we present as follows. Consider edges $v_0 v_i$

$$\begin{aligned} \gamma(v_0 v_i) - (\gamma(v_0) + \gamma(v_i)) &= 4k + 3i - (1 + 3i) \\ &= 4k - 1 \\ &= \tau_2. \end{aligned}$$

For edges $v_0 v_i^j, j = 1,3$

$$\begin{aligned} &\gamma(v_0 v_i^j) - \left(\gamma(v_0) + \gamma(v_i^j)\right) \\ &= 4k + 3i + j - 2 - (1 - 3i + j - 2) \\ &= 4k - 1 \\ &= \tau_2. \end{aligned}$$

For edges $v_i^j v_i^{j+1}$, we provide below.

k	i	j	$\gamma(v_i^j v_i^{j+1}) - \left(\gamma(v_i^j) + \gamma(v_i^{j+1})\right)$
even	$1 \le i \le \frac{k}{2}$	1,2	$4k - 1 = \tau_2$
	$\frac{k}{2} + 1 \le i \le k$	1,2	$2(k-1) = \tau_1$
	$1 \le i \le \lfloor \frac{k}{2} \rfloor$	1	$4k - 1 = \tau_2$
odd	$i = \lfloor \frac{k}{2} \rfloor + 1$	1	$4k - 1 = \tau_2$
	$\lceil \frac{k}{2} \rceil$	2	$2(k-1) = \tau_1$
	$\lceil \frac{k}{2} \rceil + 1 \le i \le k$	1,2	$2(k-1) = \tau_1$

Based on the label of vertices and the edges, it is clear that each edge $e = uw \in E$, the value of $\gamma(uw) - (\gamma(u) + \gamma(w))$ is among the different constants τ_1 and τ_2. The vertices get label forms a set $\{1, 2, 3, \ldots, 4k + 1\}$, thus, we conclude that the hibiscus graph admits a reverse super edge bimagic labeling. Then the hibiscus graph become a reverse super edge bimagic graph. ■

Here, we give an illustration of reverse super edge bimagic labeling of hibiscus graph $H_f^{(4)}$ and $H_f^{(5)}$ in Fig. 7 and Fig. 8 respectively.

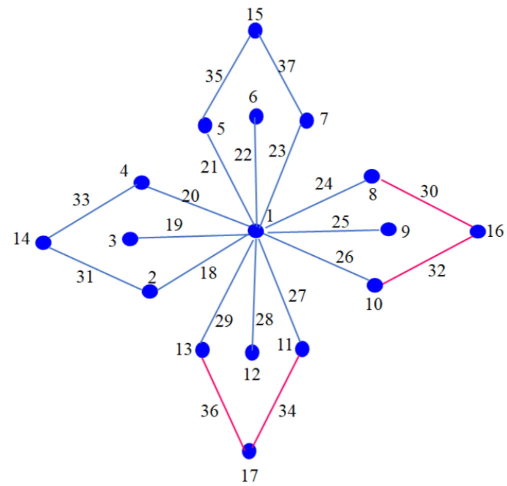

Fig. 7: Reverse super edge bimagic labeling of hibiscus graph $H_f^{(4)}$ with $\tau_1 = 6$ and $\tau_2 = 15$

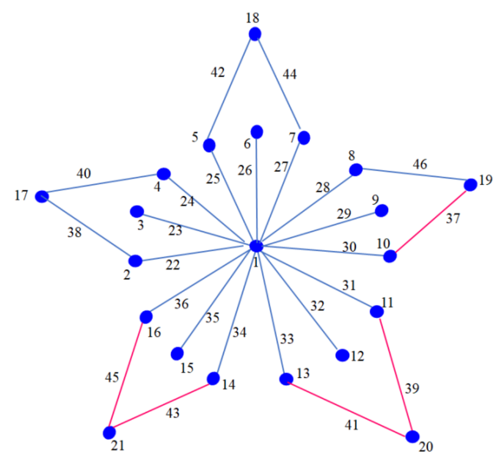

Fig. 8: Reverse super edge bimagic labeling of hibiscus graph $H_f^{(4)}$ with $\tau_1 = 8$ and $\tau_2 = 19$

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