Fuzzy Amicable sets of an Almost Distributive Fuzzy Lattice

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***Abstract*—In this paper, we introduce the concept of Fuzzy Amicable sets, we prove some properties of Fuzzy Amicable set, too. We also prove that two Fuzzy compatible elements of an Almost distributive Fuzzy Lattice (ADFL) are equal if and only if their corresponding unique Fuzzy amicable elements are equal. We define the homomorphism of two Almost Distributive Fuzzy lattices(ADFL) and finally we observe that any two Fuzzy amicable set in an Almost Distributive Fuzzy Lattice (ADFL) are isomorphic.**

***Index Terms*—Almost distributive fuzzy lattice (ADFL), almost distributive lattice (ADL), fuzzy amicable Set, fuzzy compatible set, M-fuzzy amicable element.**

I. INTRODUCTION

THE axiomatization of Boole's two valued propositional calculus led to the concept of Boolean Algebra and the class of Boolean Algebras (Ring). This includes the ring theoretic generalizations and the lattice theoretic generalizations like Heyting Algebras and distributive lattice. U.M. Samy and G.C. Rao [1] introduced the concepts of an ADL as a common abstraction of distributive lattice.

On the other hand, Zadeh [2] was the first mathematician who introduced the concepts of fuzzy and to define and study fuzzy relations, Sanchez [3] and Goguen [4] was adapted this concept. The notion of partial order and lattice order goes back to 19th century investigations in logic. The concepts of fuzzy sublattices and fuzzy ideals of a lattice was introduced by Yuan and Wu [5]. Fuzzy lattice was defined as a fuzzy algebra by Ajmal and Thomas [6] and they characterized fuzzy sublattices as a first time. In 2000, fuzzy ideal and fuzzy filters of a lattice was defined and, characterized in terms of join and meet operations by Attallah [7]. In 2009, fuzzy partial order relation was characterized in terms of its level set by Chon [8]. Chon in the same paper defined a fuzzy lattice as a fuzzy relation, developed basic properties and characterized a fuzzy lattice by its level set. As a continuation of these studies, in 2016 Berhanu et al. [9] define an Almost Distributive Fuzzy Lattice as a generalization of Fuzzy Lattice and fuzzify some properties of the classical Almost Distributive Lattice using the fuzzy partial order relation and fuzzy lattice defined by Chon [8]. As a contination of Berhanu et al. [9], In this work we introduce a new Mathematical notion, Fuzzy amicable sets and Fuzzy compatible sets of an Almost Distributive Fuzzy Lattice that preserve different properties of classical amicable set and compatible sets of an Almost Distributive Lattice. In addition, homomorphisms of Almost Distributive Fuzzy Lattice is introduced. In this work, we will use Fuzzy Lattice defined by Chon [8], and an Almost Distributive Fuzzy Lattice introduced by Berhanu et al. [9].

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II. PRELIMINARIES

In this section we give some basic preliminary concepts of An Almost Distributive lattice , and Almost Distributive Fuzzy lattices, such as Definitions, Lemmas, Corollary's and Theorems from previous results of different scholars on the area.

Definition 1 ([9]): Let (R,∨,∧,0) be an algebra of type (2,2,0) and (R,A) be a fuzzy poset.Then we call $L=(R,A)$ is an Almost Distributive Fuzzy Lattice (ADFL) if the following axioms are satisfied:

1) $A(a,a\vee 0)=A(a\vee 0,a)=1$,
2) $A(0,0\wedge a)=A(0\wedge a,0)=1$,
3) $A((a\vee b)\wedge c,(a\wedge c)\vee(b\wedge c))=A((a\wedge c)\vee(b\wedge c),(a\vee b)\wedge c)=1$,
4) $A(a\wedge(b\vee c),(a\wedge b)\vee(a\wedge c))=A((a\wedge b)\vee(a\wedge c),a\wedge(b\vee c))=1$,
5) $A(a\vee(b\wedge c),(a\vee b)\wedge(a\vee c))=A((a\vee b)\wedge(a\vee c),a\vee(b\wedge c))=1$,
6) $A((a\vee b)\wedge b,b)=A(b,(a\vee b)\wedge b)=1$.

Theorem 2 ([9]): Let (R, A) be an ADFL . Then a = b ⇔ A(a, b) = A(b, a) = 1.

Definition 3 ([9]): Let (R, A) be an ADFL . Then for any a, b ∈ R, a ≤ b if and only if A(a, b) > 0 .

Theorem 4 ([9]): If (R, A) is an ADFL then a ∧ b = a if and only if A(a, b) > 0.

Lemma 5 ([9]): Let (R, A) be an ADFL and a, b ∈ R such that a ≠ b. If A(a, b) > 0 then A(b, a) = 0.

Lemma 6 ([9]): Let (R, A) be an ADFL. Then for each a and b in R,

1) A(a∧b, b) > 0 and A(b∧a, a) > 0.
2) A(a, a∨b) > 0 and A(b, b∨a) > 0.

Lemma 7 ([9]): Let (R, A) be an ADFL.For any a and b in R, A(a∧b, b∧a) = 1 whenever A(a, b) > 0.

Now we give some basic results of an ADFL.

Lemma 8 ([9]): Let (R, A) be an ADFL . Then for any a and b in R, we have

1) A(0, a∧0) > 0;
2) A(a, a∧a) > 0;
3) A((a∧b)∨b, b) = 1;
4) A(a∨a, a) > 0;
5) A(a, 0∨a) > 0.

Lemma 9 ([9]): Let (R,A) be an ADFL.Then for any a and b in R we have

1) A(a, a∧(a∨b)) = 1;
2) A(a∨(a∧b), a) = 1;
3) A(a∨(a∧b), a∧(a∨b)) = 1;
4) A(a, (a∨b)∧a) = 1;
5) A(a∨(b∧a), a) = 1;
6) A(a∨(b∧a), (a∨b)∧a) = 1.

Corollary 10 ([9]): Let (R, A) be an ADFL.Then for any a and b in R,

1) A(a∨b, a) > 0 if and only if A(b, a∧b) > 0.
2) A(a∨b, b) > 0 and A(b, a∨b) > 0 iff A(a∧b, a) > 0 and A(a, a∧b) > 0.

Theorem 11 ([9]): Let (R, A) be an ADFL.Then for any a,b in R the following are equivalent.

1) A((a∧b)∨a, a) > 0 and A(a, (a∧b)∨a) > 0.
2) A(a∧(b∨a), a) > 0 and A(a, a∧(b∨a)) > 0.
3) A(a∨b, b∨a) > 0 and A(b∨a, a∨b) > 0.
4) The infimum of a and b exists in R and equals a∧b.
5) A(a∧b, b∧a) > 0 and A(b∧a, a∧b) > 0.
6) The supremum of a and b exists in R and equals a∨b. (7) There exists x ∈ R such that A(a, x) > 0 and A(b, x) > 0.
7) A((b∧a)∨b, b) > 0 and A(b, (b∧a)∨b)> 0.
8) A(b∧(a∨b), b) > 0 and A(b, b∧(a∨b)) > 0.

Lemma 12 ([9]): Let (R, A) be an ADFL.Then for any a, b in R, we have

1) A(a∨b, a∨(b∨a)) = 1.
2) A((a∨b)∨a, a∨b) = 1.
3) A((a∨b)∨a, a∨(b∨a)) = 1.

Lemma 13 ([9]): Let (R, A) be an ADFL.Then for any a, b, c ∈ R, A((a∨b)∧c, (b∨a)∧c) > 0 and A((b∨a)∧c, (a∨b)∧c) > 0.

Lemma 14 ([9]): Let (R, A) be an ADFL.Then for any a, b, c ∈ R, A(a∧(b∧c), (a∧b)∧c) = 1.

Lemma 15 ([9]): Let (R, A) be an ADFL. Then for any a, b, c ∈ R, A(a∧b∧c, b∧a∧c) > 0 and A(b∧a∧c, a∧b∧c) > 0.

Lemma 16 ([9]): Let (R, A) be an ADFL. If $a_1,\ldots,a_n, b \in R$ and $(i_1,\ldots,i_n)$ is any permutation of (1, 2, ., ., ., n).Then A($a_{i_1}\wedge\cdots\wedge a_{i_n}\wedge b$, $a_1\wedge\cdots\wedge a_n\wedge b$) > 0 and A($a_1\wedge\cdots\wedge a_n\wedge b$, $a_{i_1}\wedge\cdots\wedge a_{i_n}\wedge b$) > 0.

Definition 17 ([9]): The fuzzy poset (R, A) is directed above if and only if the poset (R, ≤) is directed above.
Here, we characterize ADFL as DFL.

Theorem 18 ([9]): Let (R, A) be an ADFL.Then the following are equivalent.

1) (R, A) is DFL.
2) The fuzzy poset (R, A) is directed above.
3) A(a∨b, b∨a) > 0 and A(b∨a, a∨b) > 0.
4) A((a∧b)∨c, (a∨c)∧(b∨c)) > 0 and A((a∨c)∧(b∨c), (a∧b)∨c) > 0.
5) A(a∧b, b∧a) > 0 and A(b∧a, a∧b) > 0.
6) The relation θ = {(a, b) ∈ R×R | A(a, b∧a) > 0} is antisymmetric.

III. Fuzzy Amicable Set

In this section we introduce a new mathematical notion , Fuzzy Amicable Sets of an Almost Distributive Fuzzy Lattice and we investigate and prove some results.

A. Fuzzy Compatible Set of An Almost Distributive Fuzzy Lattice

Definition 19: Let *(R,A)* be an *ADFL*. For any a , $b \in R$, we say that a is fuzzy compatible with b *(written* $a \sim_A b$*)* if *A(a∧b, b∧a) > 0* and *A(b∧a, a∧b) > 0* or equivalently, *A(a∨b, b∨a) > 0* and *A(b∨a, a∨b) > 0*.
For a sub set S of R, if for all a , $b \in S$ a $\sim_A$ b, then S is said to be fuzzy compatible.
Note that a maximal set in this paper is defined in the usual sense, and next we give the definition of Fuzzy amicable sets.

Definition 20: Let *(R, A)* be an *ADFL* and let *(M, A)* be a fuzzy maximal compatible set. Then an element a of R is said to be M-fuzzy amicable if there is an element d of M such that *A(a, d∧a) > 0*. We call *(M,A)* fuzzy amicable set, if every elements of R is M-fuzzy amicable.

Example: Let $R = \{0, a, b, c\}$ and $M = \{0, a, b\}$. Define two binary operations ∨ and ∧ in R as follow:

∨	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	b
c	c	c	c	c

and

∧	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	a	b	c

Define a fuzzy relation $A : R^2 \to [0,1]$ by:
A(a, 0)=A(b, 0)=A(c, 0)=A(c, b)=A(c, a)=A(b, a)=0,
A(0, 0)=A(a, a)=A(b, b)=A(c, c)=1,*A(0, a)=A(0, b)=A(0, c)=0.6*,
A(a, b)=0.3,*A(a, c)=0.5*, and *A(b, c)=0.2*
Clearly, *(R,A)* is an *ADFL*. Now, consider $M \subseteq R$. For any x, $y \in M$, *A(x ∧ y, y ∧ x)> 0* and *A(y ∧ x, x ∧ y)> 0*. Hence *(M, A)* is a fuzzy compatible set, and it is Fuzzy maximal compatible set. Thus, *(M, A)* is a fuzzy maximal compatible set. Now, let $x \in R$. Then there exists y in M such that *A(x, y ∧ x)>0*. Hence, every elements of R is M-fuzzy amicable. Therefore, *(M, A)* is fuzzy amicable set.

Lemma 21: Let *(R, A)* be an *ADFL*, and let *(M, A)* be a fuzzy maximal set in *(R, A)* and $x \in R$ be such that $x \sim_A a$ for all $a \in M$. Then $x \in M$.

Proof: Let *(M, A)* be a fuzzy maximal set in *(R, A)*, and let $x \in R$ such that $x \sim_A a$, for all $a \in M$.
Assume $M' = M \cup \{x\}$. Since $a \sim_A x$ for all $a \in M$ and *(M, A)* is a fuzzy compatible set, (M', A) is fuzzy compatible set such that $M \subseteq M'$. Since *(M, A)* is fuzzy maximal compatible set in *(R, A)*, $M = M' \cup \{x\}$.
Hence x belongs to M. ■

Proposition 22: Let *(R, A)* be an *ADFL* and let *(M, A)* be a fuzzy maximal set in *(R, A)*. Let $a \in M$. Then for any $x \in R$, $x \wedge a \in M$.

Proof: Let *(M, A)* be a fuzzy maximal set in an *ADFL (R, A)* and let $a \in M$. Let $x \in R$ and let b be any arbitrary elements of M. Then $A((x\wedge a)\wedge b, b\wedge(x\wedge a)) = A(x\wedge a\wedge b, b\wedge x\wedge a) = A(x\wedge b\wedge a, b\wedge x\wedge a) = A(b\wedge x\wedge a, b\wedge x\wedge a)$, Since $A(x\wedge b\wedge a, b\wedge x\wedge a) > 0$ and $A(b\wedge x\wedge a, x\wedge b\wedge a) > 0$. Hence $A(x\wedge b\wedge a, b\wedge x\wedge a) > 0$.

Since $A(b\wedge x\wedge a, b\wedge x\wedge a) > 0$, $A((x\wedge a)\wedge b, b\wedge(x\wedge a)) > 0$. Similarly, $A(b\wedge(x\wedge a),(x\wedge a)\wedge b) > 0$. Hence $x\wedge a$ belongs to M. ■

Corollary 23: Let *(M, A)* be a fuzzy maximal compatible set in an *ADFL (R, A)*. Then, for any $x \in R$ *and* $a \in M$, $A(x, a) > 0$ implies $A(x \wedge b, b \wedge x) > 0$ and $A(b \wedge x, x \wedge b) > 0$ for every $b \in M$.

Proof: Let *(M, A)* be a fuzzy maximal compatible set in an *ADFL (R, A)*. Suppose $A(x, a) > 0$, where $x \in R$ and $a \in M$. Let $b \in M$. Then $A(x\wedge b, b\wedge x) = A(x\wedge a\wedge b, b\wedge x) = A(x\wedge b\wedge a, b\wedge x) = A(b\wedge x\wedge a, b\wedge x) = A(b\wedge x, b\wedge x) > 0$,
Similarly, $A(b\wedge x, b\wedge x) > 0$. ■

Lemma 24: Let *(M, A)* be a fuzzy maximal compatible set in an *ADFL (R, A)* and a be an M-fuzzy amicable element. Then, there exists an element d of M such that $A(a, d\wedge a) > 0$ and if $e \sim_A d$ and $A(a, e\wedge a) > 0$, then $A(d, e) > 0$. Thus, if *(M, A)* is a fuzzy amicable set, then to every $a \in R$ there exists $a_A \in M$ such that, for every $x \in M, A(x, a_A) > 0$ and $A(a_A, x) > 0$ if and only if $A(a, x\wedge a) > 0$ and $A(x, a\wedge x) > 0$ and hence, given a and M, such that a_A is unique.

Proof: Let *(M, A)* be a fuzzy maximal set in an *ADFL (R, A)* and let a be an *M-fuzzy amicable* element. since a is an *M- fuzzy amicable*, there exists be $\in M$ such that $A(a, b \wedge a) > 0$. Hence by Proposition 22, $a \wedge b \in M$. Let $d = a \wedge b$. Since $A(d, a \wedge b) = A(a \wedge b, d) = 1$, $d \in M$.
$A(a, d \wedge a) > 0$, since $A(a, d\wedge a) = A(a, a\wedge b\wedge a) = A(a, a\wedge a) = A(a, a) > 0$.

Suppose $e \sim_A d$ and $A(a, e \wedge a) > 0$. Then $A(d, e) = A(a\wedge b, e) = A(e\wedge a\wedge b, e) = A(e\wedge d, e) = A(d\wedge e, e) > 0$ Hence $A(d, e) > 0$. Thus, $d = a \wedge b$ is the smallest element of M with this properties.
For every M-fuzzy amicable a of R, write $d = a_A$ where $d = a \wedge b$, $b \in M$.
since, a is M-fuzzy amicable, there exists $a_A \in M$ such that $A(a, a_A\wedge a) > 0$ and if $e \sim_A a_A$ and $A(a, e\wedge a) > 0$, then $A(a_A, e) > 0$.
Suppose *(M, A)* is a fuzzy amicable set. Hence, every elements of R is M-fuzzy amicable. Let $a \in R$.. Then there exists a smallest $a_A \in M$ such that $A(a, a_A\wedge a) > 0$. Let $x \in M$. Suppose $A(x, a_A) > 0$ and $A(a_A, x) > 0$. Now, we show that $A(a, x\wedge a) > 0$ and $A(x, a\wedge x) > 0$ $A(a, x\wedge a) = A(a_A\wedge a, x\wedge a) = A(x\wedge a, x\wedge a) = 1 > 0$ Hence, $A(a, x\wedge a) > 0$. Similarly, $A(x, a\wedge x) = A(a_A, a\wedge x) = A(a\wedge b, a\wedge x) = A(a\wedge a\wedge b, a\wedge x) = A(a\wedge a_A, a\wedge x) = A(a\wedge x, a\wedge x) > 0$ Hence, $A(x, a\wedge x) > 0$
Conversely, suppose $x \in M$ such that $A(a, x\wedge a) > 0$ and $A(x, a\wedge x) > 0$.
Since a_A and x are both elements of M, $a_A \sim_A x$ and by assumption we have $A(a, x\wedge a) > 0$.
Hence by the first argument of this Lemma, we have $A(a_A, x) > 0$.
Similarly, $A(x, a_A) = A(a\wedge x, a_A) = A(a_A\wedge a\wedge x, a_A) = A(a\wedge a_A\wedge x, a_A) = A(a\wedge x\wedge a_A, a_A) = A(x\wedge a_A, a_A) > 0$ Hence $A(x, a_A) > 0$. Therefore, for any $a \in R$, there exists a unique $a_A \in M$ such that $A(a, a_A\wedge a) > 0$ and if $e \sim_A a_A$ and $A(a, e\wedge a) > 0$, $A(a_A, e) > 0$, ■

Theorem 25: Let *(M, A)* be a fuzzy amicable set of an ADFL (R, A). Then for any x, y $\in$ R,

1) $A((x\vee y)_A, x_A\vee y_A) = A(x_A\vee y_A, (x\vee y)_A) > 0$
2) $A((x\wedge y)_A, x_A\wedge y_A) = A(x_A\wedge y_A, (x\wedge y)_A) > 0$

Proof: let (M, A) be a fuzzy amicable set in an ADFL (R, A) and let x, y $\in$ R.

1. Since (M, A) is a fuzzy amicable set and $(x\vee y) \in R$, by Lemma 24, there exists a unique $(x\vee y)_A$ in M such that $(A(x\vee y, (x\vee y)_A\wedge(x\vee y)) > 0$ and $A((x\vee y)_A, (x\vee y)\wedge(x\vee y)_A) > 0$. On the other hand, $(A(x\vee y, (x_A\vee y_A)\wedge(x\vee y)) > 0$ and $A(x_A\vee y_A, (x\vee y)\wedge(x_A\vee y_A)) > 0$, since $A(x\vee y, (x_A\vee y_A)\wedge(x\vee y)) = A((x\vee(y_A\wedge x))\vee((x_A\wedge y)\vee y), (x_A\vee y_A)\wedge(x\vee y)) = A(((x_A\wedge x)\vee(y_A\wedge y))\vee((x_A\wedge y)\vee(y_A\wedge y)), (x_A\vee y_A)\wedge(x\vee y)) = A(((x_A\vee y_A)\wedge x)\vee((x_A\vee y_A)\wedge y), (x_A\vee y_A)\wedge(x\vee y)) = A((x_A\vee y_A)\wedge(x\vee y), (x_A\vee y_A)\wedge(x\vee y)) > 0$. Hence $A(x\vee y, (x_A\vee y_A)\wedge(x\vee y)) > 0$, and $A(x_A\vee y_A, (x\vee y)\wedge(x_A\vee y_A)) = A(x_A\vee y_A, ((x\vee y)\wedge x_A)\vee((x\vee y)\wedge y_A) = A(x_A\vee y_A, ((x\wedge x_A)\vee(y\wedge x_A))\vee((x\wedge y_A)\vee(y\wedge y_A))) = A(x_A\vee y_A, (x_A\vee(y\wedge x_A)\vee((x\wedge y_A)\vee y_A)) = A(x_A\vee y_A, x_A\vee y_A) > 0$. Hence, $A(x_A\vee y_A, (x\vee y)\wedge(x_A\vee y_A)) > 0$.
Thus, by Lemma 24, $A(x_A\vee y_A, (x\vee y)_A) > 0$ and $A((x\vee y)_A, x_A\vee y_A) > 0$. 2. Since (M, A) is a fuzzy amicable set and $(x\wedge y) \in R$, by Lemma 24, there exists a unique $(x\wedge y)_A \in M$ such that $A((x\wedge y), (x\wedge y)_A\wedge(x\wedge y)) > 0$ and $A((x\wedge y)_A, (x\wedge y)\wedge(x\wedge y)_A) > 0$. On the other hand, $A(x\wedge y, (x_A\wedge y_A)\wedge(x\wedge y)) > 0$ and $A(x_A\wedge y_A, (x\wedge y)\wedge(x_A\wedge y_A)) > 0$, since $A(x\wedge y, (x_A\wedge y_A)\wedge(x\wedge y)) = A(x\wedge y, x_A\wedge x\wedge y_A\wedge y) = A(x\wedge y, x\wedge y) > 0$, and similarly, $A(x\wedge y, (x_A\wedge y_A)\wedge(x\wedge y)) = A(x\wedge y, x_A\wedge y_A\wedge x\wedge y) = A(x\wedge y, x_A\wedge y_A\wedge x\wedge y) = A(x\wedge y, x_A\wedge x\wedge y_A\wedge y) = A(x\wedge y, x_A\wedge y_A) > 0$.
Hence, $A(x\wedge y, (x_A\wedge y_A)\wedge(x\wedge y)) > 0$, and it implies that $A((x\wedge y)_A, x_A\wedge y_A) > 0$ and $A((x_A\wedge y_A, (x_\wedge y)_A) > 0$. ■

Proposition 26: Let (M, A) be a fuzzy maximal set in an ADFL (R, A) and let x, y $\in$ R be M-fuzzy amicable and x $\sim_A$ y. Then, $A(x_A, y_A) > 0$ and $A(y_A, x_A) > 0$ if and only if $A(x, y) > 0$ and $A(y, x) > 0$.

Proof: Let (M, A) be a fuzzy maximal set and let x, y $\in$ R be M-fuzzy amicable such that x $\sim_A$ y. Suppose $A(x_A, y_A) > 0$ and $A(y_A, x_A) > 0$.
$A(x, x\wedge y) = A(x, y\wedge x) = A(x, y_A\wedge y\wedge x) = A(x, y\wedge y_A\wedge x) = A(x, y_A\wedge x) = A(x, x_A\wedge x) = A(x, x)$, and similarly; $A(y, x\wedge y) = A(y, x_A\wedge x\wedge y) = A(y, x\wedge x_A\wedge y) = A(y, x_A\wedge x\wedge y) = A(y, x_A\wedge y) = A(y, y_A\wedge y) = A(y, y)$. Hence, A(x, x$\wedge$y) > 0 and A(y, x$\wedge$y)> 0. Now,
$A(x, y) \geq Sup_{r\in R} min(A(x, r), A(r, y)) \geq min(A(x, x\wedge y), A(x\wedge y, y)) > 0$.
Similarly, $A(y, x) \geq Sup_{r\in R} min(A(y, r), A(r, x)) \geq min(A(y, x\wedge y), A(x\wedge y, x)) > 0 = min(A(y, x\wedge y), A(y\wedge x, x) > 0) > 0$.
Hence, A(x, y) > 0 and A(y, x)> 0.
Conversely,assume $A(x, y)$ ¿ $0 and A(y, x)$ ¿ $0. Then A(x_A, y_A) = A(x\wedge x_A, y_A) = A(y\wedge x_A, y_A) = A(y_A\wedge y\wedge x_A, y_A) = A(y\wedge y_A\wedge x_A, y_A) = A(y_A\wedge x_A, y_A) = A(x_A\wedge y_A, y_A) > 0$ Hence A(x_A, y_A)> 0 and similarly, $A(y_A, x_A) = A(y\wedge y_A, x_A) = A(x\wedge y_A, x_A) = A(x_A\wedge x\wedge y_A, y_A) = A(x\wedge x_A\wedge y_A, x_A) = A(x_A\wedge y_A, x_A) = A(y_A\wedge x_A, x_A) > 0$. Thus, $A(x_A, y_A) > 0$ and $A(x_A, y_A) > 0$. ■

B. Homomorphism on Fuzzy Amicable Sets

In this section, we define homomorphism of ADFLs as follows and we also prove the isomorphism of Fuzzy amicable sets.

Definition 27: Let L = (R, A) and K = (M, B) be two ADFL's and let f be a map from L to K. Then f is side to be homomorphism from an ADFL L to an ADFL K if the following axiom holds true:

1) $f(x\wedge_R y) = f(x)\wedge_M f(y)$ for all x, y $\in$ R.
2) $f(x\vee_R y) = f(x)\vee_M f(y)$ for all x, y $\in$ R.
3) $f(0_R) = 0_M$ where 0_R and 0_M are the zeros of R and M, respectively.

A homomorphism *f* from L to K is called epimorphism, if *f* is an on-to map from L to K.
A homomorphism *f* from L to K is called monomorphism, if *f* is a one-to-one map from L to K.
A homomorphism *f* from L to K is called isomorphism, if *f* is both on -to and one-to-one map from L to K.
A homomorphism *f* is called automorphism, if *f* is isomorphism on L.

Definition 28: Let *L = (R, A)* and *K = (M, B)* be two *ADFL's* and let *f* be a homomorphism from *L* to *K*. The kernel of *f* is defined as follow: $ker f = \{x\in R/A(f(x), 0_M) > 0\}$.

Lemma 29: Let f be a homomorphism from an ADFL L = (R, A) to an ADFL K = (M, B). For any x, y $\in$ R, B(f(x), f(y)) >0 whenever A(x, y) > 0. If B(f(x), f(y)) > 0 and f is monomorphism, then A(x, y) > 0 for x, y $\in$ R.

Proof: Let *f* be homomorphism from *L* to *K* and let *x, y* $\in$ *R*. suppose *A(x, y) >0*. Then $x\wedge_R y = x$ and it follows that $f(x\wedge_R y) = f(x)$. Consequently, $f(x)\wedge_M f(Y) = f(x\wedge_R y) = f(x)$, and it follows $f(x)\wedge_M f(Y) = f(x)$ and then $f(x)\leq f(y)$. Hence, *B(f(x), f(y)) > 0*.
Suppose *f* is monomorphism and *B(f(x), f(y)) > 0*, where *x, y*$\in$ *R*. Then f(x)$\wedge_M$f(y) = f(x).
Since f is homomorphism, $f(x\wedge_R y) = f(x)\wedge_M f(y) = f(x)$.
$\Rightarrow f(x\wedge_R y) = f(x)$, and it follows $x\wedge_R y = x$. Hence, *A(x, y) > 0*, since $x\leq y$. ■

Corollary 30: Let f be an on to map from an *ADFL (R, A)* to an *ADFL (H, B)*. If f preserves order, then f is a homomorphism.

Proof: Suppose f is an on to map which preserves order. Let $x, y\in R$ such that $x\leq y$. Then $f(x)\leq f(y)$ and hence $f(x)\wedge_H f(y) = f(x)$ and $f(x)\vee_H f(y) = f(y)$.
Since f is well defined and $x\leq y$, $f(x\wedge_R y) = f(x)$. On the other hand $f(x)\wedge_H f(y) = f(x)$ and it implies that $f(x\wedge_R y) = f(x)\wedge_H f(y)$. Similarly, we have $f(x\vee_R y) = f(x)\vee_H f(y)$.
Finally, let 0_R and 0_H are the zero elements of (M, A) and (H, B) respectively. Since f is an onto map, there exists $t\in R$ such that $f(t) = 0_H$. Since 0_R is the zero element of (R, A), $0_R\leq t$ and since f preserves order, $f(0_R)\leq f(t) = 0_H$. on the other hand, since 0_H is the zero element of (H, B), $0_H\leq f(0_R$.
Hence, f is homomorphism. ■

Next we prove the existence of Fuzzy maximal compatible set isomorphic with a given fuzzy maximal compatible set in an ADFL.

Theorem 31: Let (M, A) be a fuzzy maximal compatible set of an ADFL L = (R, A) and $a\in R$ be M-fuzzy amicable. Then, there exists a fuzzy maximal compatible set (M', A) in L such that $a\in M'$ and the fuzzy lattice (M', A) is isomorphic with the fuzzy lattice (M, A).

Proof: Let (M, A) be a fuzzy maximal compatible set and let $a\in R$ be an M-fuzzy amicable. Define $M' = \{x\wedge(a\vee x)/x\in M\}$. Let *b, c* $\in M'$. Then, there exist x and y in M such that b = x$\wedge$(a$\vee$ x) and c = y$\wedge$(a$\vee$ y).
Now, $A(b\wedge c, c\wedge b) = A(b\wedge c, (y\wedge(a\vee y))\wedge(x\wedge(a\vee x)) = A(b\wedge c, x\wedge((y\wedge(a\vee y))\wedge(a\vee x))) = A(b\wedge c, (y\wedge x)\wedge((a\vee y)\wedge(a\vee x))) = A(b\wedge c, (y\wedge x)\wedge(a\vee(y\wedge x))) = A(b\wedge c, ((y\wedge x)\wedge a)\vee(y\wedge x)) = A(b\wedge c, ((x\wedge y\wedge a)\vee(x\wedge y)) = A(b\wedge c, (x\wedge y)\wedge(a\vee(x\wedge y))) = A(b\wedge c, (x\wedge y)\wedge((a\vee x)\wedge(a\vee y))) = A(b\wedge c, (y\wedge x)\wedge((a\vee x)\wedge(a\vee y))) = A(b\wedge c, y\wedge(x\wedge(a\vee x)\wedge(a\vee y))) = A(b\wedge c, (x\wedge(a\vee x))\wedge(y\wedge(a\vee y))) = A(b\wedge c, b\wedge c) > 0$.
Similarly, $A(c\wedge b, b\wedge c) = A(c\wedge b, c\wedge b) > 0$. Hence, ($M'$, A) is fuzzy compatible set.
Since *a* is M-fuzzy amicable, there exists an element *d* of *M* such that *A(a, d*$\wedge$ *a)> 0*. Then *a*$\wedge$ *d*$\in$ *M* and it follows that $(a\wedge d)\wedge(a\vee(a\wedge d))\in M'$, Since $A(a, (a\wedge d)\wedge(a\vee(a\wedge d))) = A(a, a\wedge(d\wedge(a\vee(a\wedge d)))) = A(a, a\wedge d\wedge a) = A(a, d\wedge a) > 0$, similarly, $A((a\wedge d)\wedge(a\vee(a\wedge d)), a) = A(d\wedge a, a) > 0$, $a = (a\wedge d)\wedge(a\vee(a\wedge d))\in M'$. Hence, $a\in M'$
Since (M', A) and (M, A) are fuzzy maximal sets, for any *a, b* $\in M'$ and *c, d* $\in$ M, $A(a\wedge b, b\wedge a) = A(b\wedge a, a\wedge b) = 1$ and $A(c\wedge d, d\wedge c) = A(d\wedge c, c\wedge d) = 1$. Thus, there exist *sup's* and *inf's* of { *a, b* } *and* { *c, d*}. Hence, (M', A) and (M, A) are fuzzy lattices.
Finally, let's define a map from (M, A)in to (M', A) by *f(x) = x*$\wedge$ *(a*$\vee$ *x)* for all *x* $\in$ *M*, and let *x, y* $\in$ *M* such that *A(x, y) > 0* and *A(y, x) > 0*. Now, $A(f(x), f(y)) = A(f(x), y\wedge(a\vee y)) = A(f(x), x\wedge(a\vee x)) = A(f(x), f(x)) > 0$, and similarly, *A(f(y), f(x)) > 0*. Hence *f* is well- defined.
Since $(M, A) and (M', A)$ are fuzzy lattices, to show f is homomorphism it suffices to show that f is an on-to map and it preserves order.
Let $b\in M'$. Then there exists $x\in M$ such that $b = x\wedge(a\vee x) = f(x)$. Hence f is an on- to map.
Let *x, y* $\in$ *M*, such that *A(x, y) > 0*. Then, since f is well-defined and $A(x,y) > 0$, $A(f(x\wedge y), f(x)) > 0$ and $A(f(x), f(x\wedge y)) > 0$. $A(f(x)\wedge f(y), f(x)) = A((x\wedge(a\vee x))\wedge(y\wedge(a\vee y)), f(x)) = A((x\wedge y)\wedge[(a\vee x)\wedge(a\vee y)], f(x)) = A((x\wedge y)\wedge[a\vee(x\wedge y)], f(x)) = A(f(x\wedge y), f(x)) > 0$. Hence, $A(f(x)\wedge f(y), f(x)) > 0$
Similarly, $A(f(x), f(x)\wedge f(y)) = A(f(x), (x\wedge(a\vee x))\wedge(y\wedge(a\vee y))) = A(f(x), (x\wedge y)\wedge((a\vee x)\wedge(a\vee y))) = A(f(x), (x\wedge y)\wedge(a\vee(x\wedge y))) = A(f(x), f(x\wedge y)) > 0$. Hence, $A(f(x), f(x)\wedge f(y)) > 0$
Hence, $f(x)\wedge f(y) = f(x)$ and it implies that $A(f(x), f(y)) > 0$.
Thus by Corollary 30, f is a homomorphism/epimorphism.
Let $x\in ker f = \{x\in R/A(f(x), 0) > 0\}$. Then, $A(f(x), 0) > 0$, and it follows that $A(x\wedge(a\vee x), 0) > 0$. Then either $A(x, 0) > 0$ or $A(a\vee x, 0) > 0$.
Suppose A(x, 0) is not greater than 0. Then $A(a\vee x, 0) > 0$ and it implies that $A(a, 0) > 0$ and $A(x, 0) > 0$, which is a contradiction.
Hence, $A(x, 0) > 0$ and it follows that x = 0.
Thus $ker f = \{0\}$. Hence f is monomorphism and it follows that f is isomorphism. ■

Next we prove that any Fuzzy amicable set is isomorphic with a Fuzzy maximal set in ADFL.

Theorem 32: Let (M, A) be a fuzzy amicable set and (M', A) be a fuzzy maximal set in an ADFL (R, A). Then the correspondence $x \longrightarrow x_A$ is a fuzzy isomorphism of (M',A) on to a fuzzy sublattice of (M, A) containing 0, where x_A denotes the unique element of M such that $A(x_A, x \wedge x_A) > 0$ and $A(x_A, x_A \wedge x) > 0$.

Proof: Let (M, A) be a fuzzy amicable set and let (M',A) be a fuzzy maximal set in (R, A).
Define f from M' in to Im f by $f(x) = x_A$ for all x $\in M'$.
Let x, y $\in M'$ such that $A(x,y) > 0$ and $A(y,x) > 0$. Since (M, A) is a fuzzy amicable set and $M' \subseteq R$, there exist x_A and y_A in M such that $A(x_A, x \wedge x_A) > 0, A(x, x_A \wedge x) > 0, A(y_A, y \wedge y_A) > 0$ and $A(y, y_A \wedge y) > 0$. Since x and y are $M-$fuzzy amicable and $x \sim_A y$, $A(x,y) > 0$ and $A(y,x) > 0$ implies $A(x_A, y_A) > 0$ and $A(y_A, x_A) > 0$. *i.e* $A(x,y) > 0$ and $A(y,x) > 0$ implies $A(f(x), f(y)) > 0$ and $A(f(y), f(x)) > 0$.
Hence f is well - defined.
To show f is homomorphism, it suffices to show f is an on to map and it preserves order.
Since we are considering Imf(the Image of f), f is an on to map from (M, A) to (Imf, A).
Let $x, y \in M'$ such that $A(x,y) > 0$. Then $x \sim_A y$ and hence $x \wedge y \in M'$. Since f is well defined and $A(x,y) > 0$, we have $A(fx \wedge y), f(x)) > 0$ and $A(f(x), f(x \wedge y)) > 0$.

$A(f(x) \wedge f(y), f(x)) = A(x_A \wedge y_A, f(x)) = A((x \wedge y)_A, f(x)) = A(f(x), f(x)) > 0$. Similarly, $A(f(x), f(x) \wedge f(y)) = A(f(x), x_A \wedge y_A) = A(f(x), (x \wedge y)_A) = A(f(x), f(x)) > 0$. Hence, $f(x) \wedge f(y) = f(x)$ and it implies that $A(f(x), f(y)) > 0$. Thus by Corollary 30, f is homomorphism/ epimorphism.
Finally, let $x, y \in M'$ such that $A(f(x), f(y)) > 0$ and $A(f(y), f(x)) > 0$. Then $(A(x_A, y_A) > 0$ and $A(y_A, x_A) > 0$.
Hence, by Proposition 30, $A(x,y) > 0$ and $A(y,x) > 0$. Thus f is fuzzy monomorphism and the result follows. ■

Theorem 33: Any two fuzzy amicable sets in an ADFL (R, A) are fuzzy isomorphic.

Proof: Let (M, A) and (M', A) are any two fuzzy amicable sets in (R, A).
Define f from (M', A) to (M, A)by $f(x) = x_A$. since (M, A) is fuzzy maximal set, by Theorem 31, f is fuzzy monomorphism.
Now we need to show that f is an on to map.
Let $a \in M$. Then $a \in A_{M'}(R)$, where $A_{M'}(R) = \{x \in R/$ there exists $x_A \in M'$ such that $x_A \wedge x = x$ and $x \wedge x_A = x_A\}$ = R. Since every elements of R are M'-fuzzy amicable, there exists $a'_A \in M'$ such that
$A(a, a'_A \wedge a) > 0$ and $A(a'_A, a \wedge a'_A) > 0$.......................... $(*)$
On the other hand, since $a'_A \in R$ and it is an M-fuzzy amicable, there exists $(a'_A)_A \in M$ such that
$A(a'_A, (a'_A)_A \wedge a'_A) > 0$ and $A((a'_A)_A, a'_A \wedge (a'_A)_A) > 0$ .... $(**)$
Hence for any $a'_A \in R$, there exists a and $(a'_A)_A$ in M such that $(*)$ and $(**)$ holds true. But by Lemma 24, the element should be unique that satisfies $(*)$ and $(**)$. Then, $a = (a'_A)_A = f(aA')$.
Hence $f(a'_A) = a$ and it follows that f is fuzzy epimorphism.
Thus, the result follows. ■

Definition 34: Let (M, A) be a fuzzy maximal compatible set of an ADFL (R, A). An upper bound of (M, A) in (R, A) is called a unielement of (M, A).

Lemma 35: If a unielement of a fuzzy maximal compatible set (M, A) exists, then it is unique and that is in M.

Proof: Let $u \in R$ be a unielement of (M, A). Then $A(a,u) > 0$, for all $a \in M$. Hence $A(a, a \wedge u) = (a \wedge u, a) = 1$. Now, $A(a \wedge u, u \wedge a) = A(a, u \wedge a) = A(a \wedge a, u \wedge a) = A(a \wedge u \wedge a, u \wedge a) = A(u \wedge a \wedge a, u \wedge a) = A(u \wedge a \wedge a, u \wedge a) > 0$. Similarly, $A(u \wedge a, a \wedge u) > 0$. Hence, $a \sim_A u$ for all $a \in M$. Thus, $u \in M$. Suppose u is not unique. Let $k \in R$ be a unielement of (M, A) such that $u \neq k$. Since k is unielement and $u \in M$, $A(u,k) > 0$. By the above argument,$k \in M$ and u is a unielement of (M, A). Hence $A(k,u) > 0$ and by antisymmetry property of A, we have $u = k$. But this is a contradiction. Thus, u is a unique element, if it exists. ■

Lemma 36: Let u be a maximal element of an ADFL (R, A) and let $M_u = \{x \in R / A(x,u) > 0\}$. Then (M_u, A) is a fuzzy maximal compatible set in (R,A) with u as its unielement.

Proof: Let $M_u = \{x \in R / A(x,u) > 0\}$, where u is a maximal element of (R, A). Let $x, y \in M$. Then $A(x,u) > 0$ and $A(y,u) > 0$. $A(x \wedge y, y \wedge x) = A(x \wedge y, y \wedge x \wedge u) = A(x \wedge y, y \wedge (x \wedge u) \wedge u) = A(x \wedge y, (x \wedge u) \wedge (y \wedge u)) = A(x \wedge y, x \wedge y)0$. Similarly, $A(y \wedge x, x \wedge y) > 0$. Thus, $x \sim_A y$, for all $x, y \in M_u$. Hence (M_u, A) is a fuzzy compatible set.
Assume there exists a fuzzy compatible set (M, A) of (R, A) with unielement u such that $M_u \subseteq M$. Let $a \in M$. Then $A(a,u) > 0$, since u is a unielement of (M, A). Hence $a \in M_u$ and it follows that $M \subseteq M_u$.
Thus, (M_u, A) is a fuzzy maximal set with unielement u. ■

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