Comparison of Numerical Methods on Pricing of European Put Options

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***Abstract*—Put option is a contract to sell some underlying assets in the future with a certain price. On European put options, selling only can be exercised at maturity date. Behavior of European put options price can be modeled by using the Black-Scholes model which provide an analytical solution. Numerical approximation such as binomial tree, explicit and implicit finite difference methods also can be used to solve Black-Scholes model. Some numerical methods are applied and compared with the analytical solution to determine the best numerical method. The results show that numerical approximations using the binomial tree is more accurate than explicit and implicit finite difference method in pricing European put options. Moreover when the value of T is higher then the error obtained is also higher, while the error obtained is lower when the value of N is higher. The value of T and N cause the increase of the computation time. When the value of T is higher the computation time is lower, while computation time is higher if the value of N is higher. Overall, the lowest computation time is obtained by using an explicit finite difference method with an exceptional as the value of T is big and the value of N is small. The lowest computation time is obtained by using a binomial tree method.**

***Index Terms*—European options, finite difference, binomial, Black Scholes.**

I. INTRODUCTION

An option is a contract giving rights to its holder to buy or to sell, and it depends on certain conditions and a certain time [1]. An American option is an option which can be exercised at any time until maturity date while a European option is an option which only can be exercised at maturity date. A European option can be divided into two types, i.e. call option and put option. A European call option gives the holder right to buy underlying assets while a European put option gives the holder right to sell it.

The Black-Scholes model [1] is a model of options pricing asset on underlying assets [2]. The model is widely used because of its simplicity in obtaining the option price analytically. There are many ways to solve the Black-Scholes model, such as a binomial tree method, an implicit and explicit finite difference method.

A binomial tree method is first introduced by Cox et al (1979), [3], [4] to obtain price option. The method is known as a simple and efficient method to provide the option pricing. A finite difference method is a popular method to provide a numerical solution of differential equations [5]. There are two finite difference methods used to obtain the option price, i.e. finite difference method using an explicit scheme and an implicit scheme. Some studies about the use of a finite difference method in obtaining the price of option are in the following. Lateef and Verma used time fractional for pricing European call option [6] for investigating the stability of the time discretization. The result shows that there was not significant change when the time fractional is changed. It was quite different from Zhang et al work [2], they solve time fractional Black-Scholes method by using an implicit discrete scheme. Experimental data show that the result was close to the analytical result.

In this research, we compare three methods for pricing a European put option. The three methods used are binomial tree method, implicit and explicit finite difference method. The results will be compared with pricing a European put option solved analytically.

Manuscript received December 15, 2017; accepted May 6, 2019.

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II. MODELS AND PRELIMINARIES

A. Black-Scholes Differential Equations

Let the underlying assets of options follow a geometric Brownian motion,

$$dS = \mu S dt + \sigma S dW \tag{1}$$

where S is the price stock, dW is the Wiener Process, μ is the price stock rate expectation and σ is the price stock movement level.

Let $V(S,t)$ be the option price which depends on stock price S and time t. By using Ito lemma, then (1) becomes

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(\mu S dt + \sigma S dW) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt \tag{2}$$

Let a portfolio consists of one option contract and a number of stock and is defined as

$$\Pi = V - \Delta S \tag{3}$$

with $\Delta = \partial V/\partial S$. From (3), the change of portfolio value is written as

$$d\Pi = dV - \Delta dS \tag{4}$$

The change of the portfolio value due to a riskless interest rate can be stated as

$$d\Pi = r\Pi dt \tag{5}$$

By substituting (2) and (4) into (5), the Black-Scholes partial differential equation can be obtained as follows.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{6}$$

B. A Binomial Tree Method [7]

A European put option is a financial contract giving holder rights to sell stocks at maturity date t with strike price K. The payoff of a put option is defined as

$$V = [K - S]^+ = \max\{K - S, 0\} \quad (7)$$

where $V = K - S$ as $K - S > 0$ and $V = 0$ as $K - S < 0$.

The option price is determined based on a no-arbitrage argument by having a current stock price S_0 and option price V_0. The option will be exercised at maturity time T. During the life of option, the stock price can either increase or decrease. The movement of stock price is driven by an increasing factor (u) or a decreasing factor (d). The movement of stock price at (S_n) time step and possible m values is written as,

$$S_n^m = d^{m-n} u^n S_0, \qquad n = 0, 1, ..., m \quad (8)$$

Aligned to the movement of stock price S, the payoffs of the option in (7) are $V_n(u)$ and $V_n(d)$ when the stock price increases and decreases respectively.

A riskless portfolio consists of a long position in Δ shares and a short position in one option. The value of option at maturity due to the stock price movement or payoff is

$$V_n = \begin{cases} V_n(u) = \max\{K - uS_0, 0\}, \text{if } S \text{ up} \\ V_n(d) = \max\{K - dS_0, 0\}, \text{if } S \text{ down} \end{cases}$$

The value of portfolio is set up as the difference between the value of stock price at particular time node in (8) and the payoff in the related node. Subsequently, the values of portfolio for up and down movement are equal so that Δ in (9) can be seen as the ratio of the change in option price to the change in stock price for between nodes movement.

$$\Delta = \frac{V_n(u) - V_n(d)}{S_0 u - S_0 d} \quad (9)$$

Taking the present value of the portfolio in (9) and setting up the cost of portfolio, we obtain

$$V = e^{-rT}[p f_u + (1-p) f_d] \quad (10)$$

where

$$p = \frac{e^{rT} - d}{u - d} \quad (11)$$

For a general n-step, (10) will be computed recursively to the current time step $t = 0$.

C. Finite Difference Method

In the finite difference method, discretization of the domain is conducted as follows

$$\begin{aligned} \frac{\partial V}{\partial t} &\approx \frac{V_{i,j+1} - V_{i,j}}{\delta t} && (12) \\ \frac{\partial V}{\partial s} &\approx \frac{V_{i+1,j} - V_{i-1,j}}{2\delta s} && (13) \\ \frac{\partial^2 V}{\partial s^2} &\approx \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\delta s)^2} && (14) \end{aligned}$$

The time discretization is using the forward scheme and the spatial discretization is using the central difference scheme.

1) Explicit Finite Difference Method: The explicit finite difference method, using the discretization in (12) and substituting into (6) will results in

$$\begin{aligned} V_{i,j+1} = {} & \left(v_1 a_{i,j} - \frac{1}{2} v_2 b_{i,j}\right) V_{i-1,j} + && (15) \\ & (1 - 2v_1 a_{i,j} + \delta t c_{i,j}) V_{i,j} + && (16) \\ & \left(v_1 a_{i,j} + \frac{1}{2} v_2 b_{i,j}\right) V_{i+1,j} + && (17) \\ & O\left((\delta t)^2, \delta t (\delta S)^2\right) && (18) \end{aligned}$$

where $v_1 = \dfrac{\delta t}{(\delta S)^2}$ and $v_2 = \dfrac{\delta t}{\delta S}$. Moreover,

$$\begin{aligned} a_{i,j} &= -\frac{1}{2}\sigma^2 (i\,\delta S)^2 && (19) \\ b_{i,j} &= -r\,i\,\delta S && (20) \\ c_{i,j} &= r. && (21) \end{aligned}$$

2) Implicit Finite Difference Method: Using the same discretization in (12) and substituting it into (6), the implicit finite difference formula can be obtained as follows,

$$\begin{aligned} V_{i,j} = {} & \left(-v_1 a_{i,j+1} - \frac{1}{2} v_2 b_{i,j+1}\right) V_{i-1,j} + && (22) \\ & \left(1 + 2v_1 a_{i,j+1} - \delta t c_{i,j+1}\right) V_{i,j+1} + && (23) \\ & \left(-v_1 a_{i,j+1} + \frac{1}{2} v_2 b_{i,j+1}\right) V_{i+1,j+1} + && (24) \\ & O\left((\delta t)^2, \delta t (\delta S)^2\right) && (25) \end{aligned}$$

where $v_1 = -\dfrac{\delta t}{(\delta S)^2}$ and $v_2 = \dfrac{\delta t}{\delta S}$. Moreover,

$$\begin{aligned} a_{i,j+1} &= -\frac{1}{2}\sigma^2 (i\,\delta S)^2 && (26) \\ b_{i,j+1} &= -r\,i\,\delta S && (27) \\ c_{i,j+1} &= r. && (28) \end{aligned}$$

D. The Analytical Solution

An analytical solution of the Black-Scholes differential equation as shown in (6) can be obtained by Fourier transformation. The analytical solution is

$$\begin{aligned} V(S(t),t) &= KC(x,t^*) \\ &= Ke^x N(d_1) - e^{-kt^*} N(d_1) \\ &= S(t)N(d_1) - exp\left(\frac{-r}{\frac{1}{2}\sigma^2} \frac{1}{2}\sigma^2 (T-t)\right) N(d_2) \\ &= S(t)N(d_1) - e^{r(T-t)} N(d_2) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S(t)/K) + \frac{1}{2}\sigma^2 (T-t)\left(\frac{r}{\frac{1}{2}\sigma^2} + 1\right)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log(S(t)/K) + \frac{1}{2}\sigma^2 (T-t)\left(\frac{r}{\frac{1}{2}\sigma^2} - 1\right)}{\sigma\sqrt{T-t}} \end{aligned}$$

III. Results and Discussions

As an illustration in calculating a European put option, some parameters used are listed as follows,

$$\sigma = 0.2; T = 3; M = 30; N = 75; S_{\max} = 150$$

$S_0 = 10, 20, \ldots, 100$; $r = 0.03, 0.05, 0.07$; $K = 30, 50, 70$;

where S is stock price, r is interest level and K is strike price.

Various parameters are used to see the influence of interest level and strike price on the behavior of a put option value. The value of the European option for various interest rate are presented in Fig. 1, Fig. 2 and Fig. 3 for strike price $K = 30$, $K = 50$, and $K = 70$ respectively.

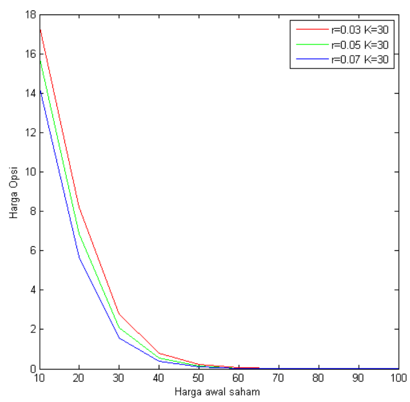

Fig. 1. Graphic of comparison toward a European put option movement with different r and $K = 30$.

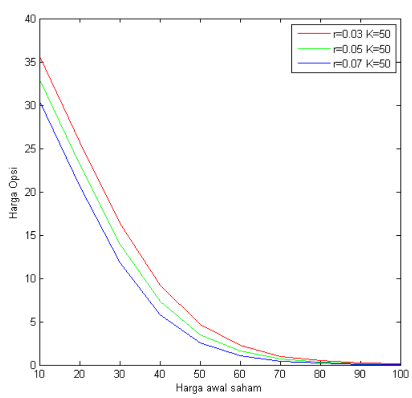

Fig. 2. Graphic of comparison toward a European put option movement with different r and $K = 50$.

From Fig. 1, 2, and 3, it can be seen that the various value of S_t influence to price of European put option. The higher S_t given, the lower price of European put option. Also, it can be seen that the different value of interest level, i.e. $r = 0.03, 0.05, 0.07$ with each strike price $K = 30, 50, 70$ influence price of European put option. The simulation results show that the higher interest level r, the lower price of European put option. Next, for various parameters K, the results are given in Fig. 4, 5, and 6.

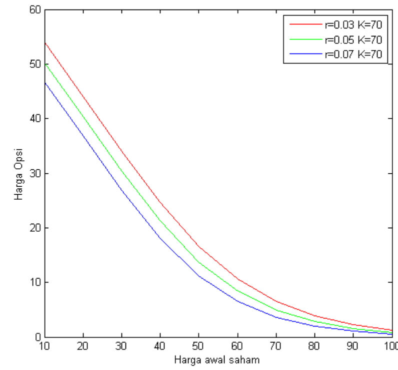

Fig. 3. Graphic of comparison toward a European put option movement with different r and $K = 70$

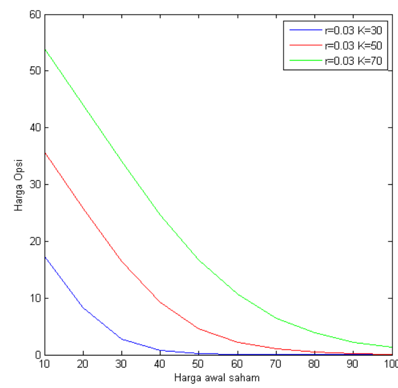

Fig. 4. Graphic of comparison toward a European put option movement with $r = 0.03$ and different K.

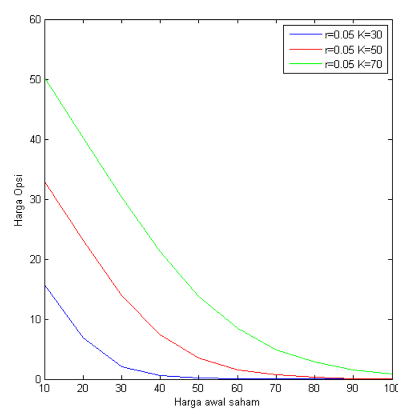

Fig. 5. Graphic of comparison toward a European put option movement with $r = 0.05$ and different K.

From the simulation results with different values of strike price $K = 30$, $K = 50$ and $K = 70$ as shown in Fig. 4, 5, 6, it is clear that the strike price r influences price of European put option for fixed r.

When discussing about the numerical method, it cannot be independent from error. In numerical calculation to price of the European put option, we use three different methods such

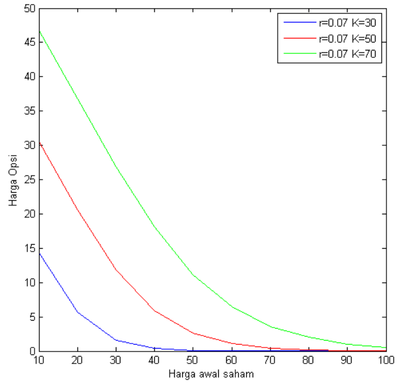

Fig. 6. Graphic of comparison toward a European put option movement with $r = 0.07$ and different K.

as binomial tree method, explicit and implicit finite difference methods. The results are compared with the analytical calculation of them.

TABLE I
RESULT AND ERROR COMPARISON OF A PUT OPTION VALUE WITH DIFFERENT METHODS AT $T = 3$ AND $N = 75$.

S	Numerical Method			Analytical Solution	Error		
	Binomial	Explicit	Implicit		Binomial	Explicit	Implicit
10	17.4205	17.4268	17.4302	17.4209	0.0004	0.0059	0.0093
20	8.2058	8.171	8.1809	8.2095	0.0037	0.0385	0.0286
30	2.7789	2.6763	2.6617	2.7878	0.0089	0.1115	0.1261
40	0.8056	0.7446	0.7385	0.7946	0.011	0.05	0.0561
50	0.2125	0.2	0.2016	0.2132	0.0007	0.0132	0.0116
60	0.0571	0.0545	0.0571	0.057	1E-04	0.0025	1E-04
70	0.015	0.0154	0.017	0.0156	0.0006	0.0002	0.0014
80	0.0041	0.0045	0.0054	0.0044	0.0003	1E-04	0.001
90	0.0012	0.0014	0.0018	0.0013	0.0001	0.0001	0.0005
100	0.00034973	0.00043469	0.00063718	0.0004	0.00005027	0.00003469	0.00023718
TC	0.004749	0.035485	0.045862				

From Table I, numerical calculation results using binomial tree method give better results than that of using explicit and implicit finite difference methods. In this case, maturity date given is $T = 3$ and the number of discretization is $N = 75$. While the value of N is increasing, the obtained error is decreasing.

TABLE II
RESULT AND ERROR COMPARISON OF A PUT OPTION VALUE WITH DIFFERENT METHODS AT $T = 3$ AND $N = 4500$

S	Numerical Method			Analytical Solution	Error		
	Binomial	Explicit	Implicit		Binomial	Explicit	Implicit
10	17.4209	17.4289	17.4282	17.4209	0	0.008	0.0073
20	8.2095	8.1759	8.1760	8.2095	0	0.0336	0.0035
30	2.7877	2.6691	2.6689	2.7878	0.0001	0.1187	0.01189
40	0.7947	0.7416	0.7415	0.7946	0.0001	0.053	0.0531
50	0.2132	0.2008	0.2008	0.2132	0	0.0124	0.0124
60	0.0570	0.0558	0.0558	0.0570	0	0.0012	0.0012
70	0.0156	0.0162	0.0162	0.0156	0	0.0006	0.0006
80	0.0044	0.0049	0.0050	0.0044	0	0.0005	0.0006
90	0.0013	0.0016	0.0016	0.0013	0	0.0003	0.0003
100	0.0003994	0.00053184	0.00053522	0.0004	6e-07	0.000132	0.000135
TC	0.759954	0.068252	0.130408				

By using $T = 3$ and $N = 4500$, the numerical calculation results using binomial tree method is still better than that of using explicit and implicit finite difference methods. While the value of N is increasing, the obtained error is decreasing. Overall, this case is also satisfied by using explicit and implicit finite different methods. ˜

For $T = 10$ and $N = 75$, the explicit finite difference method obtain greater error value as shown at table III. The error value obtained by using binomial tree method is lower than that of by using implicit finite difference method. ˜

TABLE III
RESULT AND ERROR COMPARISON OF A PUT OPTION VALUE WITH DIFFERENT METHODS AT $T = 10$ AND $N = 75$

S	Numerical Method			Analytical Solution	Error		
	Binomial	Explicit	Implicit		Binomial	Explicit	Implicit
10	12.6661	5.9179e+09	12.6994	12.6734	0.0073	5.92e+09	0.026
20	6.4441	7.3355e+14	6.3697	6.4219	0.0222	7.34e+14	0.0522
30	3.3176	1.012e+19	3.2146	3.2783	0.0393	1.01e+19	0.0637
40	1.7738	3.3715e+22	1.6919	1.7400	0.0338	3.37e+22	0.0481
50	0.9672	3.8422e+25	0.9322	0.9634	0.0038	3.84e+25	0.0312
60	0.5666	1.8248e+28	0.5359	0.5548	0.0118	1.82e+28	0.0189
70	0.3375	4.081e+30	0.3197	0.3308	0.0067	4.08e+30	0.0111
80	0.2074	4.6528e+32	0.1970	0.2034	0.004	4.65e+32	0.0064
90	0.1278	2.8400e+34	0.1246	0.1285	0.0007	2.84e+34	0.0039
100	0.0861	9.5289e+35	0.0803	0.0832	0.0029	9.53e+35	0.0029
TC	0.003679	0.033761	0.52142				

TABLE IV
RESULT AND ERROR COMPARISON OF A PUT OPTION VALUE WITH DIFFERENT METHODS AT $T = 10$ AND $N = 4500$

S	Numerical Method			Analytical Solution	Error		
	Binomial	Explicit	Implicit		Binomial	Explicit	Implicit
10	12.6734	12.6982	12.6789	12.6734	0	0.0248	0.0055
20	6.4224	6.3654	6.3644	6.4219	0.005	0.0565	0.0475
30	3.2783	3.2231	3.2226	3.2783	0	0.0552	0.0557
40	1.7404	1.7018	1.7014	1.7400	0.0004	0.0382	0.0386
50	0.9636	0.9393	0.9391	0.9634	0.0002	0.0241	0.0243
60	0.5550	0.5399	0.5398	0.5548	0.0002	0.0149	0.0150
70	0.3309	0.3216	0.3215	0.3308	0.0001	0.0092	0.0093
80	0.2034	0.1974	0.1974	0.2034	0	0.006	0.006
90	0.1286	0.1242	0.1242	0.1285	0.0001	0.0043	0.0043
100	0.0832	0.0794	0.0795	0.0832	0	0.0038	0.0037
TC	0.744687	0.064586	0.175977				

From the result tables presented before it can be seen that the change value of T influences to error, except special case in explicit finite difference method. The higher value of T, the higher obtained error while the greater N, the lower the error. Overall, calculation price of European put option using binomial tree method gives better results than that of using explicit finite difference methods.

Beside the error, the computation time to calculate the price of a European put option is presented in the last row of all tables. The value of T and N influence to time computation. The results show that the higher T, the lower computation time while the higher N, the higher computation time. The best computation time as $N = 75$ is the binomial tree method while the best computation time as $N = 4500$ is the explicit finite difference method.

IV. CONCLUSIONS

Based on result and discussion above, it can be concluded that the higher S_0 and interest level r given, causes the lower European put option price while the higher strike price K, the higher the price of the European put option. For the effect of maturity date T and number of discretization N, the higher value of T, the higher error while the higher value of N, the lower error is. Overall, a numerical solution using binomial tree method is more accurate than that of using explicit and implicit finite difference methods. The best computation time as $N = 75$ is the binomial tree method while the best computation time as $N = 4500$ is the explicit finite difference method.

REFERENCES

[1] F. Black and M. Scholes, “The Pricing of Options and Corporate Liabilities,” *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1973.

[2] H. Zhang, F. Liu, I. Turner, and Q. Yang, "Numerical solution of the time fractional black–scholes model governing european options," *Computers & Mathematics with Applications*, vol. 71, no. 9, pp. 1772–1783, 2016.

[3] J. C. Cox, S. A. Ross, and M. Rubinstein, "Option pricing: A simplified approach," *Journal of financial Economics*, vol. 7, no. 3, pp. 229–263, 1979.

[4] P. Wilmott, S. Howison, and J. Dewynne, *The Mathematics of Financial Derivatives: A Student Introduction*, first edition ed. Cambridge University Press,, 1995.

[5] D. G. Duffy, *Transform Methods for Solving Partial Differential Equations*, 2nd ed. Chapman and Hall/CRC, 2004.

[6] A. Lateef and C. Verma, "An analysis of fractional explicit method over black scholes method for pricing option," *Global Journal of Pure and Applied Mathematics*, vol. 13, no. 9, pp. 5851–5869, 2017.

[7] J. C. Hull, *Options, Futures, and Other Derivatives*, 7th ed. Pearson Prentice Hall, 2009.