Max Plus Algebra and Petri Net Application on Scheduling of Ship Engine Component's Spare Part Ordering

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***Abstract*—Shipping company is a company that runs its business by operating the ships or other businesses that are closely related to the ship. A ship has a main engine and some auxiliary engines to support the ship performance. It needs to do maintenance of engines so that the ship can operate properly. This engine maintenance is replacement of the old engine components with the new ones if the running hours of the components are over. Therefore, in the ship, the spare parts must always be available at least one for each engine component. During this time, the company has experienced a difficulty in determining the time of spare part ordering. When the running hours of engine components are over, the spare parts were not yet available. Then, Petri Net and Max Plus Algebra model will be built to schedule the ordering of ship engine component's spare part based on the ordering flow and the running hours of engine components. The Petri Net based on the Max Plus Algebra obtains maximum time to order the spare part so that it produces the ship engine component's spare part ordering schedule in running hour form and date. Therefore, spare part of each ship engine component is always available so that the installation can be timely and never be late.**

***Index Terms*—Max-plus algebra, Petri nets, Scheduling of spare part ordering.**

I. Introduction

MAX plus algebra is the useful approach to represent the discrete event systems. This approach makes us possible to determine and analyze various kinds of system properties. Therefore, the model of these ones will be linear over max plus algebra. But in conventional algebra, it is not linear. We can analyze the systems in max plus algebra easier and simpler than the conventional systems because of this linearity [1]. One of applications of max plus algebra is a scheduling of crystal sugar production system [2].

A Petri net is a mathematical modeling tool which can be applied to represent the state evolution of the discrete event systems. Petri net is called autonomous if every transition in this Petri net has at least an input state. This means that there is no transition which is enabled without any condition. In other words, autonomous Petri net does not have a transition which is always enabled [3]. Timed Petri net is an extension of Petri net. In this paper, we use the autonomous timed Petri net and max plus algebra model for scheduling of ship engine components' spare part ordering. Furthermore, we will build a model of Max Plus Algebra using Supply Chain model to obtain the date of spare part ordering. For more detailed discussion of supply chain using max plus algebra, the interested reader is referred to [4].

Shipping company is a company that runs its business by operating the ships or other businesses that are closely related to the ship. The ship becomes a very important part in this company. Therefore, the company must maintain the performance of the ship so that operations run optimally. The most important thing in keeping the performance of the ship is to make sure all of the ship's engine run properly so as not to cause delays in shipping time. Inside the ship there are two large groups of machines, the main engine and auxiliary engine. In order for ship engines to function normally, the system required periodic maintenance. Thus, the ship's engine does not get a breakdown. Periodic maintenance is generally in the form of checking up the replacement of components in the engine according to running hours of the components. Therefore, in the ship must always be available at least one each of component parts for the engine, so that when it is needed in the periodic maintenance, the spare parts can be used directly without disturbing the shipping schedule.

The spare part of each ship engine component is ordered from various suppliers. When the running hours of ship engine components will end, the ship crew will start to require the spare parts. Furthermore, the request will be processed by purchasing division until the spare parts are ready to be sent to the warehouse. Purchasing division will inform the spare parts requirement to suppliers who already have a working relationship with the company. The suppliers will offer the spare parts requested by different price, time availability, and quality. From some offers, purchasing division makes a summary of the offers which will be submitted to the ship manager to determine which supplier will be chosen. Following an agreement, the chosen supplier will provide the spare parts and send to the company's warehouse in a given time period. Thus, the spare part is available and ready to be sent to the ship.

During this time the company experienced difficulties in determining the time of spare parts ordering. When running hours of engine components are over, the spare parts are not available. As a result, the spare parts must be ordered from within the country or abroad and delivered by plane. Thus, the purchase cost is increasing. Meanwhile, the company expects the spare parts are always available before running hours of ship engine components end so that the ship can operate

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optimally. In addition, the purchase cost of spare parts become cheaper if ordering in the right time and not in a hurry because it can be delivered by land or sea transportation. Therefore, Petri Net and Max Plus Algebra model will be built to schedule the ordering of ship engine component's spare part based on the ordering flow and the running hour of engine components. It is expected that the spare part of each ship engine component is always available so that the installation can be timely.

II. Preliminaries

A. Max Plus Algebra

Given $\mathbb{R}_\varepsilon \overset{\text{def}}{=} \mathbb{R} \cup \{\varepsilon\}$ where $\mathbb{R}$ is a set of real numbers and $\varepsilon \overset{\text{def}}{=} -\infty$. In $\mathbb{R}_\varepsilon$, two operations are defined by:

$$x \oplus y \overset{\text{def}}{=} \max\{x,y\} \quad \text{and} \quad x \otimes y \overset{\text{def}}{=} x+y, \quad \forall x,y \in \mathbb{R}_\varepsilon.$$

Furthermore, $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is a semiring with neutral element ε and unit element $e \overset{\text{def}}{=} 0$. For $x \in \mathbb{R}_\varepsilon$ and $n \in \mathbb{N}$ with $n \neq 0$, where $\mathbb{N}$ is the set of all positive integers, we define

$$x^{\otimes n} \overset{\text{def}}{=} \underbrace{x \otimes x \otimes \cdots \otimes x}_{n},$$

whereas for $n = 0$, it is defined by $x^{\otimes n} \overset{\text{def}}{=} e(=0)$. Therefore, for each $n \in \mathbb{N}$, $x^{\otimes n}$ is defined by

$$x^{\otimes n} \overset{\text{def}}{=} \underbrace{x \otimes x \otimes \cdots \otimes x}_{n} = n \otimes x.$$

Furthermore,

$$x^{\otimes \alpha} = \alpha \otimes x, \quad \text{for } \alpha \in \mathbb{R}.$$

Adddition and multiplication of two matrices with appropriate size over max-plus algebra are defined by

$$[A \oplus B]_{ij} = [A]_{ij} \oplus [B]_{ij}$$
$$[A \otimes B]_{ij} = \bigoplus_k ([A]_{ik} \otimes [B]_{kj})$$

where A and B are matrices of appropriate dimension. As an example, the multiplication of two matrices of size 2×2 is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \otimes e \oplus b \otimes g & a \otimes f \oplus b \otimes h \\ c \otimes e \oplus d \otimes g & c \otimes f \oplus d \otimes h \end{bmatrix}$$

and the identity matrix of size 2×2 over max plus algebra is

$$\begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}.$$

B. Petri Nets

Definition 1 ([1]): Petri net is 4-tuple (P,T,A,w) with

- P : a finite set of places, $P = \{p_1, p_2, \ldots, p_n\}$;
- T : a finite set of transitions, $T = \{t_1, t_2, \ldots, t_n\}$;
- A : a set of arcs, $A \subseteq (P \times T) \bigcup (T \times P)$;
- w : a weight function, $w \to \{1,2,3,\ldots\}$.

Definition 2 ([1]): Marking x in a Petri net is a function $x : P \to \{0,1,2,\ldots\}$.

Definition 3: Transition $t_j \in T$ in a Petri net is enabled if $x(p_i) \geq w(p_i, t_j)$, for all $p_i \in I(t_j)$.

C. Prioritized Petri Nets

Each transition in a Petri net has a priority value and denoted by π. Prioritized Petri Net is a Petri Net which prioritizes one or several transition(s) so that in certain condition, the prioritized transition is selected from a set of enabled transitions. This situation is shown in Fig. 1 and 2.

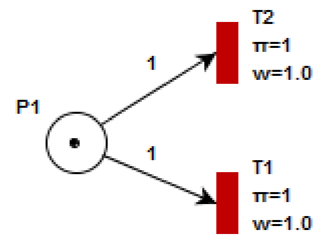

Fig. 1. Petri Net which has two transitions with the same priority.

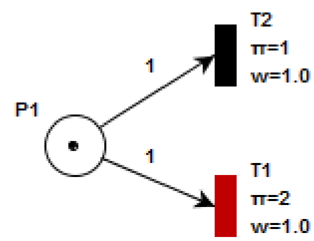

Fig. 2. Petri Net which has two transitions with different priority.

Figure 1 shows that the transition T_1 and T_2 is equally enabled and has a priority value $\pi = 1$. Meanwhile, in Fig. 2, the transition T_2 is enabled, but the transition T_1 is not enabled. This condition is caused by the priority value of transition T_2 is higher than the priority value of transition T_1. Thus, only transition T_2 can be fired.

D. System of Ship Engine Component's Spare Part Ordering

The flow of ship engine component's spare part ordering is described in Fig. 3. Furthermore, the process of ship engine

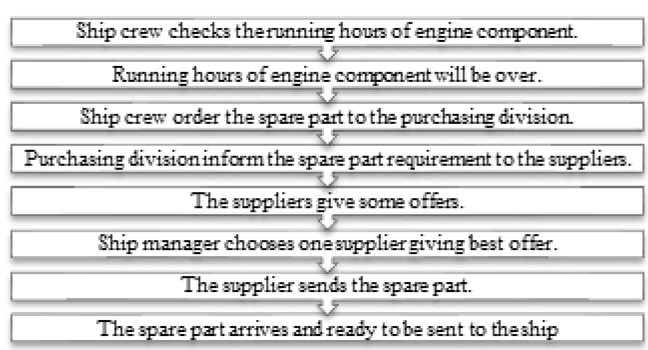

Fig. 3. Flow Chart of Ship Engine Component's Spare Part Ordering.

component's spare part ordering will be represented by a Petri Net and use it to construct a Max Plus Algebra model for obtaining the maximal time of spare part ordering.

III. Results and Discussions

A. Model of Ship Engine Component' Spare Part Ordering in Max Plus Algebra

Based on Fig. 3, we obtain Petri Net on Fig. 4 that represents the process of ship engine component's spare part ordering, where the interpretation of transitions is as follows:

- T_1: spare part ordering by the ship's crew to purchasing division ($\pi = 1$).
- T_2: dissemination of ordering information by purchasing division to suppliers ($\pi = 2$).
- T_3: offering from suppliers ($\pi = 2$).
- T_4: manager checks the offering summary from suppliers ($\pi = 2$).
- T_5: the offering of type A ($\pi = 8$).
- T_6: the offering of type A is not available or ignored ($\pi = 8$).
- T_7: the offering of type B ($\pi = 7$).
- T_8: the offering of type B is not available or ignored (π=7).
- T_9: the offering of type C ($\pi = 6$).
- T_{10}: the offering of type C is not available or ignored ($\pi = 6$).
- T_{11}: the offering of type D ($\pi = 5$).
- T_{12}: the offering of type D is not available or ignored ($\pi = 4$).
- T_{13}: the offering of type A is picked ($\pi = 3$).
- T_{14}: the offer other than type A is picked ($\pi = 2$).
- T_{15}: the offer rejected ($\pi = 2$).
- T_{16}: manager instruct the re-dissemination of information offering by purchasing division to other suppliers ($\pi = 2$).
- T_{17}: spare parts start to order ($\pi = 2$).
- T_{18}: spare parts arrived at the warehouse ($\pi = 2$).

The meaning of each place is given by

- P_1: purchasing division receives spare parts ordering
- P_2: suppliers receive ordering information from purchasing division
- P_3: the offering summary is submitted to the manager by purchasing division
- P_4: the offering of type A
- P_5: the offering of type B
- P_6: the offering of type C
- P_7: the offering of type D
- P_8: a decision on the acceptance or rejection of the offering
- P_9: saving temporary offering summary that is rejected
- P_{10}: the offering of type A is prioritized
- P_{11}: selected one supplier
- P_{12}: selected suppliers provide spare parts ordered

Furthermore, this Petri Net is used to make the following Max Plus Algebra model which produces maximum time of ship engine component's spare part ordering.

$$T_1(k) = v_{T_1,k} \otimes T_1(k-1) \quad (1)$$
$$T_2(k) = v_{T_2,k} \otimes T_1(k) \oplus v_{T_2,k} \otimes T_{16}(k-1) \quad (2)$$
$$T_3(k) = v_{T_3,k} \otimes T_2(k) \quad (3)$$
$$T_4(k) = v_{T_4,k} \otimes T_3(k) \quad (4)$$
$$T_5(k) = v_{T_5,k} \otimes T_4(k) \quad (5)$$
$$T_6(k) = v_{T_6,k} \otimes T_4(k) \quad (6)$$
$$T_7(k) = v_{T_7,k} \otimes T_4(k) \quad (7)$$
$$T_8(k) = v_{T_8,k} \otimes T_4(k) \quad (8)$$
$$T_9(k) = v_{T_9,k} \otimes T_4(k) \quad (9)$$
$$T_{10}(k) = v_{T_{10},k} \otimes T_4(k) \quad (10)$$
$$T_{11}(k) = v_{T_{11},k} \otimes T_4(k) \quad (11)$$
$$T_{12}(k) = v_{T_{12},k} \otimes T_4(k) \quad (12)$$
$$T_{13}(k) = v_{T_{13},k} \otimes [T_5(k) \oplus T_7(k) \oplus T_9(k) \oplus T_{11}(k)] \quad (13)$$
$$T_{14}(k) = v_{T_{14},k} \otimes [T_5(k) \oplus T_7(k) \oplus T_9(k) \oplus T_{11}(k)] \quad (14)$$
$$T_{15}(k) = v_{T_{15},k} \otimes [T_5(k) \oplus T_7(k) \oplus T_9(k) \oplus T_{11}(k)] \quad (15)$$
$$T_{16}(k) = v_{T_{16},k} \otimes T_{15}(k) \quad (16)$$
$$T_{17}(k) = v_{T_{17},k} \otimes [T_{13}(k) \oplus T_{14}(k)] \quad (17)$$
$$T_{18}(k) = v_{T_{18},k} \otimes T_{17}(k) \quad (18)$$

Based on equation (1), (16), and (18), we obtain

$$\begin{bmatrix} T_1(k) \\ T_{16}(k) \\ T_{18}(k) \end{bmatrix} = \begin{bmatrix} v_{T_1,k} & \varepsilon & \varepsilon \\ a & b & \varepsilon \\ c_n & d_n & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} T_1(k-1) \\ T_{16}(k-1) \\ T_{18}(k-1) \end{bmatrix} \quad (19)$$

where

- $v_{T_1,k}$: duration of spare part offering by crew that is accepted by purchasing division
- $v_{T_2,k}$: duration of dissemination offering process from purchasing division to suppliers
- $v_{T_3,k}$: duration of offering from the first supplier since receiving the request information
- $v_{T_4,k}$: duration of offering from the second supplier since receiving the request information
- $v_{T_5,k}$: duration of offering from the third supplier since receiving the request information
- $v_{T_6,k}$: duration of offering from the fourth supplier since receiving the request information
- $v_{T_7,k}$: duration submission of offering summary to the manager
- $v_{T_8,k}$: duration of the manager determines all offerings being rejected
- $v_{T_9,k}$: duration of offering from other suppliers since receiving the information
- $v_{T_{10},k}$: duration of submitting a new offering summary to manager
- $v_{T_{11},k}$: duration of the manager determines one of the offerings received
- $v_{T_{12},k}$: duration of spare parts began to be ordered since supplier determination
- $v_{T_{12},k}$: duration of spare parts began to be ordered since the determination of selected suppliers
- $v_{T_{13},k}$: duration of spare parts arrived in the warehouse since it was ordered.

and

$$a = v_{T_1,k} \otimes v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_5,k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \oplus$$
$$v_{T_1,k} \otimes v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_7,k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \oplus$$
$$v_{T_1,k} \otimes v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_9,k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \oplus$$
$$v_{T_1,k} \otimes v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_{11},k} \otimes v_{T_{15},k} \otimes v_{T_{16},k}$$

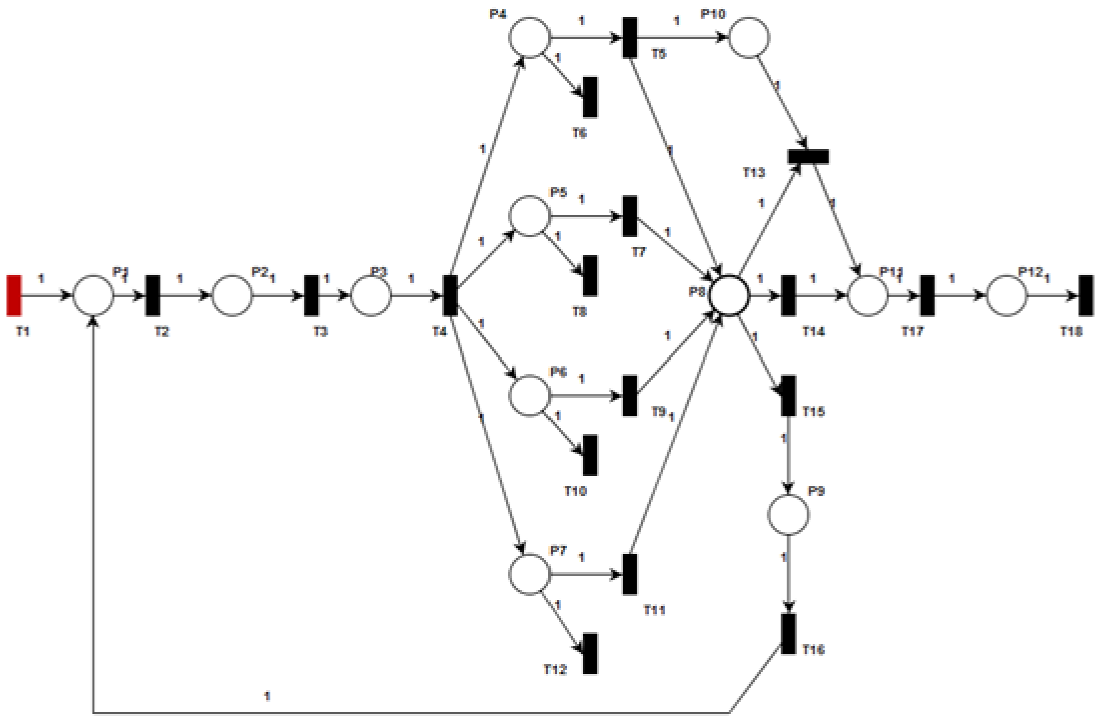

Fig. 4. Prioritized Petri Net of Engine Component's Spare Part Ordering.

$$\begin{aligned}
b = {} & v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_5,k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \oplus \\
& v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_7,k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \oplus \\
& v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_9,k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \oplus \\
& v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_{11},k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \\
c_n = {} & v_{T_1,k} \otimes \left(v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_5,k}\right)^{\otimes n} \otimes \\
& v_{T_{14},k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \otimes v_{T_{17},k} \otimes v_{T_{18},k} \oplus \\
& v_{T_1,k} \otimes \left(v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_7,k}\right)^{\otimes n} \otimes \\
& v_{T_{14},k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \otimes v_{T_{17},k} \otimes v_{T_{18},k} \oplus \\
& v_{T_1,k} \otimes \left(v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_9,k}\right)^{\otimes n} \otimes \\
& v_{T_{14},k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \otimes v_{T_{17},k} \otimes v_{T_{18},k} \oplus \\
& v_{T_1,k} \otimes \left(v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_{11},k}\right)^{\otimes n} \otimes \\
& v_{T_{14},k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \otimes v_{T_{17},k} \otimes v_{T_{18},k} \\
d_n = {} & \left(v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_5,k}\right)^{\otimes n} \otimes \\
& v_{T_{14},k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \otimes v_{T_{17},k} \otimes v_{T_{18},k} \oplus \\
& \left(v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_7,k}\right)^{\otimes n} \otimes \\
& v_{T_{14},k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \otimes v_{T_{17},k} \otimes v_{T_{18},k} \oplus \\
& \left(v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_9,k}\right)^{\otimes n} \otimes \\
& v_{T_{14},k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \otimes v_{T_{17},k} \otimes v_{T_{18},k} \oplus \\
& \left(v_{T_2,k} \otimes v_{T_3,k} \otimes v_{T_4,k} \otimes v_{T_{11},k}\right)^{\otimes n} \otimes \\
& v_{T_{14},k} \otimes v_{T_{15},k} \otimes v_{T_{16},k} \otimes v_{T_{17},k} \otimes v_{T_{18},k}
\end{aligned}$$

and n is the number of spare part requirement information spreading to the suppliers. Therefore, for $n = 2$, we obtained

$$c_2 = 39 \otimes v_{T_{18},k}$$

$$d_2 = 38 \otimes v_{T_{18},k}$$

The maximal time of spare part ordering from equation (19) is in day unit so it has to be converted to hour unit by multiplying it with 10 hours which is the average of ship's running hours every day, then we get the ordering duration. To compute the time when the spare part starts to be ordered, we have to subtract the interval for overhaul by the ordering duration and the result is showed in Table I.

Furthermore, we will construct a model of Max Plus Algebra using Supply Chain model to obtain the date of spare part ordering. Figure 5 represents the process of spare part ordering using a Supply Chain model.

Based on Fig. 5, we get the following Max Plus Algebra model:

$$\begin{aligned}
t_2(k) &= w_a \otimes t_3(k-n) \oplus t_1(k) \\
t_3(k) &= w_b \otimes t_2(k) \\
y(k) &= t_3(k)
\end{aligned}$$

where $n = 1$ is the number of spare parts.

The model can be changed into matrix equations as follow.

$$\begin{aligned}
X(k) &= A_0 \otimes X(k) \oplus A_1 \otimes X(k-n) \oplus B_0 \otimes U(k) \\
Y(k) &= C \otimes X(k)
\end{aligned}$$

where

$$U(k) = [t_1(k)]$$

$$X(k) = \begin{bmatrix} t_2(k) \\ t_3(k) \end{bmatrix}$$

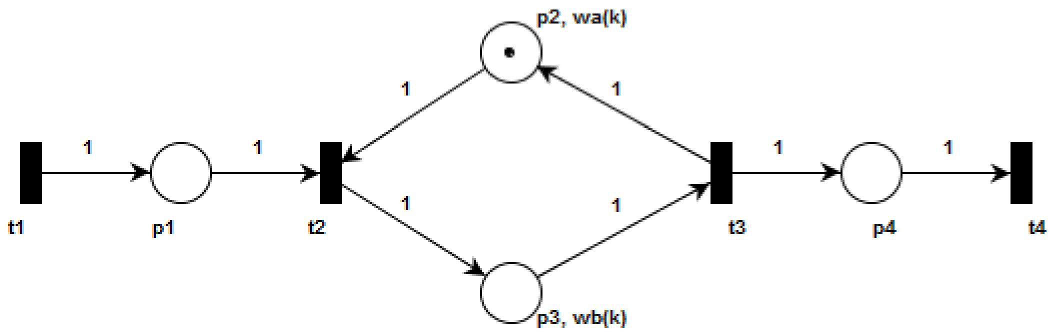

Fig. 5. Petri Net of Spare Part Ordering Using Supply Chain Model.

$$Y(k) = [t_4(k)]$$
$$A_0 = \begin{bmatrix} \varepsilon & \varepsilon \\ w_b & \varepsilon \end{bmatrix}$$
$$A_1 = \begin{bmatrix} \varepsilon & w_a \\ \varepsilon & \varepsilon \end{bmatrix}$$
$$B_0 = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix}$$
$$C = \begin{bmatrix} \varepsilon & 0 \end{bmatrix}$$

Since $A = A_0{}^* \otimes A_1$ and $B = A_0{}^* \otimes B_0$ [3], then

$$\begin{aligned} A &= A_0{}^* \otimes A_1 \\ &= \begin{bmatrix} 0 & \varepsilon \\ w_b & 0 \end{bmatrix} \otimes \begin{bmatrix} \varepsilon & w_a \\ \varepsilon & \varepsilon \end{bmatrix} \\ &= \begin{bmatrix} 0 \otimes \varepsilon \oplus \varepsilon \otimes \varepsilon & 0 \otimes w_a \oplus \varepsilon \otimes \varepsilon \\ w_b \otimes \varepsilon \oplus 0 \otimes \varepsilon & w_b \otimes w_a \oplus 0 \otimes \varepsilon \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon & w_a \\ \varepsilon & w_b \otimes w_a \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon & w_a \\ \varepsilon & w_a w_b \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} B &= A_0{}^* \otimes B_0 \\ &= \begin{bmatrix} 0 & \varepsilon \\ w_b & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \\ &= \begin{bmatrix} 0 \otimes 0 \oplus \varepsilon \otimes \varepsilon \\ w_b \otimes 0 \oplus 0 \otimes \varepsilon \end{bmatrix} \\ &= \begin{bmatrix} 0 \oplus \varepsilon \\ w_b \oplus \varepsilon \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ w_b \end{bmatrix} \end{aligned}$$

so that

$$\begin{aligned} CB &= \begin{bmatrix} \varepsilon & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ w_b \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon \otimes 0 \oplus 0 \otimes w_b \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon \oplus w_b \end{bmatrix} \\ &= \begin{bmatrix} w_b \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} CAB &= \begin{bmatrix} \varepsilon & 0 \end{bmatrix} \otimes \begin{bmatrix} \varepsilon & w_a \\ \varepsilon & w_a w_b \end{bmatrix} \otimes \begin{bmatrix} 0 \\ w_b \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon \otimes \varepsilon \oplus 0 \otimes \varepsilon & \varepsilon \otimes w_a \oplus 0 \otimes w_a w_b \end{bmatrix} \otimes \begin{bmatrix} 0 \\ w_b \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon \oplus \varepsilon & \varepsilon \oplus w_a w_b \end{bmatrix} \otimes \begin{bmatrix} 0 \\ w_b \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon & w_a w_b \end{bmatrix} \otimes \begin{bmatrix} 0 \\ w_b \end{bmatrix} \\ &= [\varepsilon \otimes 0 \oplus w_a w_b \otimes w_b] \\ &= \left[\varepsilon \oplus w_a w_b{}^{\otimes 2}\right] \\ &= \left[w_a w_b{}^{\otimes 2}\right] \end{aligned}$$

According to [1]

$$y(k) = \bigoplus_{i=0}^{\alpha} C \otimes A^{\otimes i} \otimes B \otimes u(k - i \cdot n)$$

so that for $n = 1$ and $\alpha = [k/n] = [k] = k$, we obtain

$$\begin{aligned} y(1) &= C \otimes A^{\otimes 0} \otimes B \otimes u(1 - 0 \cdot 1) \oplus C \otimes A^{\otimes 1} \otimes B \otimes u(1 - 1 \cdot 1) \\ &= C \otimes B \otimes u(1) \oplus C \otimes A^{\otimes 1} \otimes B \otimes u(1 - 1 \cdot 1) \\ &= C \otimes B \otimes u(1) \oplus C \otimes A \otimes B \otimes u(0) \end{aligned}$$

$$\begin{aligned} y(2) &= C \otimes A^{\otimes 0} \otimes B \otimes u(2 - 0 \cdot 1) \oplus C \otimes A^{\otimes 1} \otimes B \otimes u(2 - 1 \cdot 1)) \oplus \\ &\quad C \otimes A^{\otimes 2} \otimes B \otimes u(2 - 2 \cdot 1) \\ &= C \otimes B \otimes u(2) \oplus C \otimes A^{\otimes 1} \otimes B \otimes u(1) \oplus C \otimes \ A^{\otimes 2} \otimes B \otimes u(0) \\ &= C \otimes B \otimes u(2) \oplus C \otimes A \otimes B \otimes u(1) \end{aligned}$$

Therefore, for number of ordering $l = 1, 2$, we obtain

$$Y = H \otimes U$$

where

$$Y = \begin{bmatrix} y(1) \\ y(2) \end{bmatrix},$$
$$H = \begin{bmatrix} CB & \varepsilon \\ CAB & CB \end{bmatrix},$$
$$U = \begin{bmatrix} u(1) \\ u(2) \end{bmatrix}.$$

Notice that Y is time when the running hours of engine component over and U is the time when the spare part ordering starts.

Therefore, the solution is [1]

$$U = -H^T \oplus' Y \qquad (20)$$

where

$$u(1) = \min\{y_1 - h_{1,1}, y_2 - h_{2,1}\},$$
$$u(2) = \min\{y_1 - h_{1,2}, y_2 - h_{2,2}\}.$$

B. Schedule of Engine Component's Spare Part Ordering

Based on the ordering duration and overhaul interval of each spare part which are gotten before, we can determine the running hours when the ordering is started that is shown in Table I.

Spare part's name	Ordering duration (hours)	Overhaul for interval (hours)	Running hours when the ordering is started (hours)
CRANKPIN BEARING SHELL	520	16000	**15480**
CROSSHEAD BEARING SHELL	650		**15350**
O-RING - N17M6220	940	8000	**7060**
SCRAPER RING (LOWER)	940		**7060**
SCRAPER RING (UPPER)	940		**7060**
TIGHTENING RING	940		**7060**
PISTON RING - 3169804	730	8000	**7270**
PISTON RING - 3169805	1400		**6600**
GUIDE RING	880	4000	**3120**
O-RING – 4511913	490		**3510**
O-RING – 4511912	490		**3510**
O-RING - EN17M340	730		**3270**
O-RING – 4183312	940		**3060**
O-RING - EN17M365	730		**3270**
PISTON RING	730		**3270**
SEAL RING – 4184389	730		**3270**
SEAL RING – 4184390	730		**3270**
SPACER RING	730		**3270**
O-RING – 4181145	730	8000	**7270**
O-RING – 4181146	940		**7060**
O-RING – 4183002	940		**7060**
SLIDE VALVE ASS	730		**7270**
SPINDLE GUIDE ASS	730		**7270**
SPACER RING	490	16000	**15510**
SCRAPER RING	730		**15270**
O-RING – 4181145	940		**15060**
O-RING – 4181452	730		**15270**
O-RING – 4181455	730		**15270**
SLIDE VALVE	500		**15500**
SPRING	500		**15500**
THRUST PIECE	500		**15500**
CYLINDER COMPLETE	540		**15460**

TABLE I. Running Hours of Engine Component When the Spare Part Starts to Be Ordered.

Furthermore, the running hours when the ordering is started is converted into day unit by dividing it by 10 which is the average of ship's running hour every day. By using equation (20), we obtain the date when each spare part starts to be ordered that is presented on Table II until Table V below.

Spare part's name	Date when spare part starts to be ordered	
CRANKPIN BEARING SHELL	11 January 2020	29 May 2024
CROSSHEAD BEARING SHELL	29 December 2019	16 May 2024
O-RING	02 December 2016	10 February 2019
SCRAPER RING (LOWER)	02 December 2016	10 February 2019
SCRAPER RING (UPPER)	02 December 2016	10 February 2019
TIGHTENING RING	02 December 2016	10 February 2019
PISTON RING - 3169804	13 October 2017	22 December 2019

Spare part's name	Date when spare part starts to be ordered	
PISTON RING - 3169805	07 August 2017	16 October 2019
GUIDE RING	23 August 2016	27 September 2017
O-RING - 4511913	01 October 2016	05 November 2017
O-RING - 4511912	01 October 2016	05 November 2017
O-RING - EN17M340	07 September 2016	12 October 2017
O-RING - 4183312	17 August 2016	21 September 2017
O-RING - EN17M365	07 September 2016	12 October 2017
PISTON RING	07 September 2016	12 October 2017
SEAL RING – 4184389	07 September 2016	12 October 2017
SEAL RING – 4184390	07 September 2016	12 October 2017
SPACER RING	07 September 2016	12 October 2017
O-RING - 4181145	16 January 2016	26 March 2018
O-RING - 4181146	26 December 2015	05 March 2018
O-RING - 4183002	26 December 2015	05 March 2018
SLIDE VALVE ASS	16 January 2016	26 March 2018
SPINDLE GUIDE ASS	16 January 2016	26 March 2018
SPACER RING	13 February 2016	01 July 2020
SCRAPER RING	20 January 2016	07 June 2020
O-RING - 4181145	30 December 2015	17 May 2020
O-RING - 4181452	20 January 2016	07 June 2020
O-RING - 4181455	20 January 2016	07 June 2020
SLIDE VALVE	12 February 2016	30 June 2020
SPRING	12 February 2016	30 June 2020
THRUST PIECE	12 February 2016	30 June 2020
CYLINDER COMPLETE	08 February 2016	26 June 2020

TABLE II. The date of spare part's ordering for cylinder 1.

Spare part's name	Date when spare part starts to be ordered	
CRANKPIN BEARING SHELL	23 August 2018	09 January 2023
CROSSHEAD BEARING SHELL	10 August 2018	27 December 2022
O-RING	05 May 2016	14 July 2018
SCRAPER RING (LOWER)	05 May 2016	14 July 2018
SCRAPER RING (UPPER)	05 May 2016	14 July 2018
TIGHTENING RING	05 May 2016	14 July 2018
PISTON RING - 3169804	04 August 2018	12 October 2020
PISTON RING - 3169805	29 May 2018	06 August 2020
GUIDE RING	06 July 2016	18 September 2017
O-RING - 4511913	14 August 2016	18 September 2017
O-RING - 4511912	14 August 2016	18 September 2017
O-RING - EN17M340	21 July 2016	25 August 2017
O-RING - 4183312	30 June 2016	04 August 2017
O-RING - EN17M365	21 July 2016	25 August 2017
PISTON RING	21 July 2016	25 August 2017
SEAL RING – 4184389	21 July 2016	25 August 2017
SEAL RING – 4184390	21 July 2016	25 August 2017
SPACER RING	21 July 2016	25 August 2017
O-RING - 4181145	07 January 2016	17 March 2018
O-RING - 4181146	17 December 2015	24 February 2018
O-RING - 4183002	17 December 2015	24 February 2018
SLIDE VALVE ASS	07 January 2016	17 March 2018
SPINDLE GUIDE ASS	07 January 2016	17 March 2018
SPACER RING	26 August 2018	12 January 2023
SCRAPER RING	02 August 2018	19 December 2022
O-RING - 4181145	12 July 2018	28 November 2022

Spare part's name	Date when spare part starts to be ordered	
O-RING - 4181452	02 August 2018	19 December 2022
O-RING - 4181455	02 August 2018	19 December 2022
SLIDE VALVE	25 August 2018	11 January 2023
SPRING	25 August 2018	11 January 2023
THRUST PIECE	25 August 2018	11 January 2023
CYLINDER COMPLETE	21 August 2018	07 January 2023

TABLE III. The date of spare part's ordering for cylinder 2.

Spare part's name	Date when spare part starts to be ordered	
CRANKPIN BEARING SHELL	18 January 2020	05 June 2024
CROSSHEAD BEARING SHELL	05 January 2020	23 May 2024
O-RING	25 June 2016	03 September 2018
SCRAPER RING (LOWER)	25 June 2016	03 September 2018
SCRAPER RING (UPPER)	25 June 2016	03 September 2018
TIGHTENING RING	25 June 2016	03 September 2018
PISTON RING - 3169804	16 July 2016	24 September 2018
PISTON RING - 3169805	10 May 2016	19 July 2018
GUIDE RING	09 October 2015	12 November 2016
O-RING - 4511913	17 November 2015	21 December 2016
O-RING - 4511912	17 November 2015	21 December 2016
O-RING - EN17M340	24 October 2015	27 November 2016
O-RING - 4183312	03 October 2015	06 November 2016
O-RING - EN17M365	24 October 2015	27 November 2016
PISTON RING	24 October 2015	27 November 2016
SEAL RING – 4184389	24 October 2015	27 November 2016
SEAL RING – 4184390	24 October 2015	27 November 2016
SPACER RING	24 October 2015	27 November 2016
O-RING - 4181145	07 January 2016	17 March 2018
O-RING - 4181146	17 December 2015	24 February 2018
O-RING - 4183002	17 December 2015	17 March 2018
SLIDE VALVE ASS	07 January 2016	17 March 2018
SPINDLE GUIDE ASS	07 January 2016	17 March 2018
SPACER RING	13 February 2016	01 July 2020
SCRAPER RING	20 January 2016	07 June 2020
O-RING - 4181145	30 December 2015	17 May 2020
O-RING - 4181452	20 January 2016	07 June 2020
O-RING - 4181455	20 January 2016	07 June 2020
SLIDE VALVE	12 February 2016	30 June 2020
SPRING	12 February 2016	30 June 2020
THRUST PIECE	12 February 2016	30 June 2020
CYLINDER COMPLETE	08 February 2016	26 June 2020

TABLE IV. The date of spare part's ordering for cylinder 3.

Spare part's name	Date when spare part starts to be ordered	
CRANKPIN BEARING SHELL	23 August 2018	09 January 2023
CROSSHEAD BEARING SHELL	10 August 2018	27 December 2022
O-RING	03 May 2016	12 July 2018
SCRAPER RING (LOWER)	03 May 2016	12 July 2018
SCRAPER RING (UPPER)	03 May 2016	12 July 2018
TIGHTENING RING	03 May 2016	12 July 2018
PISTON RING - 3169804	24 May 2016	02 August 2018
PISTON RING - 3169805	18 March 2016	27 May 2018
GUIDE RING	12 June 2016	17 July 2017
O-RING - 4511913	21 July 2016	25 August 2017
O-RING - 4511912	21 July 2016	25 August 2017

Spare part's name	Date when spare part starts to be ordered	
O-RING - EN17M340	27 June 2016	01 August 2017
O-RING - 4183312	06 June 2016	11 July 2017
O-RING - EN17M365	27 June 2016	01 August 2017
PISTON RING	27 June 2016	01 August 2017
SEAL RING – 4184389	27 June 2016	01 August 2017
SEAL RING – 4184390	27 June 2016	01 August 2017
SPACER RING	27 June 2016	01 August 2017
O-RING - 4181145	16 January 2016	26 March 2018
O-RING - 4181146	26 December 2015	05 March 2018
O-RING - 4183002	26 December 2015	05 March 2018
SLIDE VALVE ASS	16 January 2016	26 March 2018
SPINDLE GUIDE ASS	16 January 2016	26 March 2018
SPACER RING	26 August 2018	12 January 2023
SCRAPER RING	02 August 2018	19 December 2022
O-RING - 4181145	12 July 2018	28 November 2022
O-RING - 4181452	02 August 2018	19 December 2022
O-RING - 4181455	02 August 2018	19 December 2022
SLIDE VALVE	25 August 2018	11 January 2023
SPRING	25 August 2018	11 January 2023
THRUST PIECE	25 August 2018	11 January 2023
CYLINDER COMPLETE	21 August 2018	07 January 2023

TABLE V. The date of spare part's ordering for cylinder 4.

IV. CONCLUSIONS

Petri net can be used to represent a process of ship engine component's spare part ordering. Based on the Petri net model, we can build a max-plus-algebra model to find the maximal time of spare part ordering. Therefore, we obtain the date when the spare part should be ordered.

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