Numerical Solution of Second Order Initial Value Problems of Bratu-type Equations using Sixth Order Runge-Kutta Seven Stages Method

Hibist Bazezew Fenta and Getachew Adamu Derese

***Abstract*—In this paper, second order initial value problem of Bratu-type ordinary differential equations is solved numerically using sixth order Runge-Kutta seven stages method. The stability of the method is checked and verified. In order to justify the validity and effectiveness of the method, two model examples are solved and the numerical solutions are compared to the corresponding exact solutions. Furthermore, the results obtained using the current method are compared with the numerical results obtained by other researchers. The numerical results in terms of point-wise absolute errors presented in tables and plotted graphs show that the present method approximates the exact solutions very well.**

***Index Terms*—Bratu-type equation, second order differential equation, sixth order Runge-Kutta method, stability.**

I. Introduction

PHYSICAL situations concerned with the rate of change of one quantity with respect to another give rise to a differential equation. Differential equations are absolutely fundamental to modern science and engineering. Almost all of the known laws of physics and chemistry are actually differential equations.

Many authors have attempted to solve initial value problems (IVP) to obtain high accuracy rapidly by using numerous methods such as Taylor's method and Runge-Kutta methods. Runge-Kutta methods are more efficient than many numerical methods in terms of accuracy. The methods were developed by two German Mathematicians Carl Runge (1856-1927) and Wilhelm Kutta (1867-1944). The general Runge-Kutta methods are widely used for approximating initial value problems. In particular when the computation of higher derivatives is complicated. They are distinguished by their orders in the sense that they agree with Taylor series solution up to terms of h^r where r is the order of the method.

A sixth order Runge-Kutta method was derived by [1] depending on the fifth order Runge-Kutta method of David Goeken and Olin Johnson which needs only five function evaluations. The standard Bratu problem is used in a large variety of applications, such as the fuel ignition model of the theory of thermal combustion, the thermal reaction process model, the Chandrasekhar model of the expansion of the universe, radiated heat transfer, nanotechnology and theory of chemical reaction [2].

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According to [3], in 1850, Liouville solved the second order PDE $u_{xy} + \lambda e^u = 0$, which is associated with the Liouville model in quantum theory. In 1914, Bratu solved almost similar second order ODE $u_{xx} + \lambda e^u = 0$ which is a basic one-dimensional model in combustion theory.

Because of the mathematical and physical properties, the Bratu initial value problems have been studied extensively by many researchers, for example [4] studied a numerical solution of Bratu-type equations by using the variational iteration method; [5] considered Bratu's problems by means of modified homotopy perturbation method; [6] applied Adomian decomposition method to study the Bratu-type equations; [7] used successive differentiation method to solve Bratu equation and Bratu-type equations; [8] developed an algorithm using Runge-Kutta methods of orders four and five for first order systems of ordinary differential equations. The numerical solution of second order initial value problems of Bratu-type equations was studied by [9] using fifth order Runge-Kutta method.

The objective of the present study is to investigate numerical solutions of second order initial value problems of Bratu-type equations using sixth order Runge-Kutta seven stages method. The exact solutions will be compared with numerical solutions obtained using graphs and tables of point-wise absolute errors.

II. Formulation of the Problem

Consider the second order initial value problem of Bratu-Type equation of the form:

$$y''(x) + \lambda e^{y(x)} = g, \qquad 0 \le x \le 1 \tag{1}$$

subject to the initial conditions:

$$y(0) = \alpha, \qquad y'(0) = \beta \tag{2}$$

where λ, α, β are given constant numbers and $y(x)$ is unknown functions.

Using the substitution $z(x) = y'(x)$ and $z'(x) = y''(x)$, the given second order initial value problem of equation (1) with initial conditions (2) can be rewritten as:

$$\begin{cases} y'(x) = z(x) = F(x,y,z), & y(0) = \alpha \\ z'(x) = g(x) - \lambda e^{y(x)} = G(x,y,z), & z(0) = y'(0) = \beta \end{cases} \tag{3}$$

Dividing the interval $[0,l]$ into N equal sub intervals of mesh length h, the mesh point is given by

$$x_i = x_0 + ih, \qquad \text{for } i = 1,2,3,\ldots,N-1.$$

For the sake of simplicity, let $y(x_i)=y_i$, $z(x_i)=z_i$, $g(x_i)=g_i$, etc. Thus, at each nodal point x_i, equation (3) is written as:

$$\begin{cases} y_i'(x)=F(x_i,y_i,z_i), & y(0)=\alpha \\ z_i'(x)=G(x_i,y_i,z_i), & z(0)=\beta \end{cases} \tag{4}$$

where $G(x_i,y_i,z_i)=g_i-\lambda e^{y_i}$.

Equation (4) is a system of first order initial value problems. Thus, Runge-Kutta method of order six with seven stages can be applied to approximate the solution of the problem. According to [4], the general form of sixth order Runge-Kutta methods with seven stages is given as:

$$\begin{cases} y_{n+1}=y_n+\sum_{i=1}^{7} w_i k_i, \\ z_{n+1}=z_n+\sum_{i=1}^{7} w_i m_i \end{cases} \tag{5}$$

where

$$\begin{cases} k_i=hF(x_n+c_ih,y_n+\sum_{j=1}^{i-1} a_{ij}k_j,z_n+\sum_{j=1}^{i-1} a_{ij}m_j) \\ m_i=hG(x_n+c_ih,y_n+\sum_{j=1}^{i-1} a_{ij}k_j,z_n+\sum_{j=1}^{i-1} a_{ij}m_j) \end{cases}$$

The sixth order Runge-kutta method with seven stages to solve first order initial value problem of the form

$$\frac{dy}{dx}=f(x,y), \qquad y(x_0)=y_0 \tag{6}$$

is given by the following Butcher tabular form [10].

TABLE I
BUTCHER SIXTH ORDER METHOD WITH SEVEN STAGES

1/3	1/3						
2/3	0	2/3					
1/3	1/12	1/3	-1/12				
1/2	0	9/8	-3/8	-3/4	1/2		
1/2	9/44	-9/11	63/44	18/11	0	-16/11	
1	11/120	0	27/40	27/40	4/15	-4/15	11/120

Thus, from Table I, we have:

$$y_{n+1}=y_n+\frac{1}{120}(11k_1+81k_3+81k_4-32k_5-32k_6+11k_7), \tag{7}$$

where

$$\begin{aligned} k_1 &= hf(x_n,y_n) \\ k_2 &= hf(x_n+h/3,y_n+k_1/3) \\ k_3 &= hf(x_n+2h/3,y_n+2k_2/3) \\ k_4 &= hf(x_n+h/3,y_n+k_1/12+k_2/3-k_3/12) \\ k_5 &= hf(x_n+h/2,y_n-k_1/16+9k_2/8-3k_3/16-3k_4/8) \\ k_6 &= hf(x_n+h/2,y_n+9k_2/8-3k_3/8-3k_4/4+k_5/2) \\ k_7 &= hf(x_n+h,y_n+9k_1/44-9k_2/11+63k_3/44+18k_4/11 \\ &\quad -16k_6/11) \end{aligned}$$

Thus, to solve the system of initial value problems of equation (3), the sixth order Runge-Kutta method with seven stages can be re-written as:

$$\begin{aligned} y_{n+1} &= y_n+\frac{1}{120}(11k_1+81k_3+81k_4-32k_5-32k_6+11k_7) \\ z_{n+1} &= z_n+\frac{1}{120}(11m_1+81m_3+81m_4-32m_5-32m_6+11m_7) \end{aligned} \tag{8}$$

where

$$\begin{aligned} k_1 &= hF(x_n,y_n,z_n) \\ m_1 &= hG(x_n,y_n,z_n) \\ k_2 &= hF(x_n+h/3,y_n+k_1/3,z_n+m_1/3) \\ m_2 &= hG(x_n+h/3,y_n+k_1/3,z_n+m_1/3) \\ k_3 &= hF(x_n+2h/3,y_n+2k_2/3,z_n+2m_2/3) \\ m_3 &= hG(x_n+2h/3,y_n+2k_2/3,z_n+2m_2/3) \\ k_4 &= hG(x_n+h/3,y_n+k_1/12+k_2/3-k_3/12,z_n+m_1/12+ \\ &\quad m_2/3-m_3/12) \\ m_4 &= hG(x_n+h/3,y_n+k_1/12+k_2/3-k_3/12,z_n+m_1/12+ \\ &\quad m_2/3-m_3/12) \\ k_5 &= hF(x_n+h/2,y_n-k_1/16+9k_2/8-3k_3/16-3k_4/8, \\ &\quad z_n-m_1/16+9m_2/8-3m_3/16-3m_4/8) \\ m_5 &= hG(x_n+h/2,y_n-k_1/16+9k_2/8-3k_3/16-3k_4/8, \\ &\quad z_n-m_1/16+9m_2/8-3m_3/16-3m_4/8) \\ k_6 &= hF(x_n+h/2,y_n+9k_2/8-3k_3/8-3k_4/4+k_5/2, \\ &\quad z_n+9m_2/8-3m_3/8-3m_4/4+m_5/2) \\ m_6 &= hG(x_n+h/2,y_n+9k_2/8-3k_3/8-3k_4/4+k_5/2, \\ &\quad z_n+9m_2/8-3m_3/8-3m_4/4+m_5/2) \\ k_7 &= hF(x_n+h,y_n+9k_1/44-9k_2/11+63k_3/44+18k_4/11- \\ &\quad 16k_6/11,z_n+9m_1/44-9m_2/11+63m_3/44+ \\ &\quad 18m_4/11-16m_6/11) \\ m_7 &= hF(x_n+h,y_n+9k_1/44-9k_2/11+63k_3/44+18k_4/11- \\ &\quad 16k_6/11,z_n+9m_1/44-9m_2/11+63m_3/44+ \\ &\quad 18m_4/11-16m_6/11) \end{aligned}$$

III. Stability Analysis of the Method

Here, the second order initial value problems of Bratu-type equation of the form of equation (1) is reduced into first-order system of equations of the form (4). The second equation in equation (4) is:

$$z_i'=G(x_i,y_i,z_i), \qquad z(0)=\beta \tag{9}$$

where $G(x_i,y_i,z_i)=g_i-\lambda e^{y_i}$.

The nonlinear equation (9) can be linearized by expanding the function G in Taylor series about the point (x_0,y_0,z_0). Truncating it after the first term gives:

$$\begin{aligned} z' &= g(x_0,y_0,z_0)+(x-x_0)\frac{\partial G}{\partial x}(x_0,y_0,z_0)+ \\ &\quad (y-y_0)\frac{\partial G}{\partial y}(x_0,y_0,z_0)+(z-z_0)\frac{\partial G}{\partial z}(x_0,y_0,z_0) \end{aligned} \tag{10}$$

By differentiation rules of function of several variables, equation (10) can be written as:

$$z'=g_0-\lambda e^{y_0}+(x-x_0)(g_0'-\lambda y_0 e^{y_0})+\lambda(y_0-y)e^{y_0}=C \tag{11}$$

where $C=g_0-\lambda e^{y_0}+(x-x_0)(g_0'-\lambda y_0 e^{y_0})+\lambda(y_0-y)e^{y_0}$. Then $z'=C$ which is linear in the function of z.

Consider the test equation:

$$y'=\lambda y \tag{12}$$

Then, the solution of the test equation (12) at the n-th discrete point is:

$$y_n = y(x_n) = y(x_0)e^{\lambda nh} = y_0(e^{\lambda h})^n, \quad (13)$$

where y_0 is constant.

Now, by applying equation (7) on equation (12), we have:

$$k_1 = hf(y_n) = h\lambda y_n$$
$$k_2 = hf(y_n + \frac{1}{3}k_1) = h\lambda y_n + \frac{1}{3}(\lambda h)^2 y_n$$
$$k_3 = hf(y_n + \frac{2}{3}k_2) = h\lambda y_n + \frac{2}{3}(\lambda h)^2 y_n + \frac{2}{9}(\lambda h)^3 y_n$$
$$k_4 = hf(y_n + \frac{1}{12}k_1 + \frac{1}{3}k_2 - \frac{1}{12}k_3) = \lambda h y_n + \frac{1}{3}(\lambda h)^2 y_n + \frac{1}{18}(\lambda h)^3 y_n - \frac{1}{54}(\lambda h)^4 y_n$$
$$k_5 = hf(y_n - \frac{1}{16}k_1 + \frac{9}{8}k_2 - \frac{3}{16}k_3 - \frac{3}{8}k_4) = \lambda h y_n + \frac{1}{2}(\lambda h)^2 y_n + \frac{1}{8}(\lambda h)^3 y_n - \frac{1}{16}(\lambda h)^4 y_n + \frac{1}{144}(\lambda h)^5 y_n$$
$$k_6 = hf(y_n + \frac{9}{8}k_2 - \frac{3}{8}k_3 - \frac{3}{4}k_4 + \frac{1}{2}k_5) = \lambda h y_n + \frac{1}{2}(\lambda h)^2 y_n + \frac{1}{8}(\lambda h)^3 y_n - \frac{1}{16}(\lambda h)^4 y_n - \frac{5}{288}(\lambda h)^5 y_n + \frac{1}{288}(\lambda h)^6 y_n$$
$$k_7 = hf(y_n + \frac{9}{44}k_1 - \frac{9}{11}k_2 + \frac{63}{44}k_3 + \frac{18}{11}k_4 - \frac{16}{11}k_6) = \lambda h y_n + (\lambda h)^2 y_n + \frac{1}{2}(\lambda h)^3 y_n + \frac{5}{22}(\lambda h)^4 y_n + \frac{2}{33}(\lambda h)^5 y_n + \frac{5}{198}(\lambda h)^6 y_n + \frac{1}{198}(\lambda h)^7 y_n$$

By substituting the values of k_1 and $k_3 - k_7$ into

$$y_{n+1} = y_n + \frac{1}{120}(11k_1 + 81k_3 + 81k_4 - 32k_5 - 32k_6 + 11k_7)$$

we obtain:

$$y_{n+1} = E(\lambda h)y_n \quad (14)$$

where $E(\lambda h) = 1 + \lambda h + \frac{1}{2}(\lambda h)^2 + \frac{1}{6}(\lambda h)^3 + \frac{1}{24}(\lambda h)^4 + \frac{1}{120}(\lambda h)^5 + \frac{1}{720}(\lambda h)^6 + \frac{1}{2160}(\lambda h)^7$.

The error in numerical computation does not grow if the propagation error tends to zero or if at least bounded [10]. Now, from equation (13), it is easily observed that the exact value of $y(x_i)$ increases for the constant $\lambda > 0$ and decreases for $\lambda < 0$ with the factor of $e^{\lambda h}$. Meanwhile, from equation (14), the approximate value of $y(x_n)$ increases or decreases with the factor of $E(\lambda h)$. If $\lambda h \geq 0$, then $e^{\lambda h} \geq 1$. So, the sixth order Runge-Kutta method with seven stage is relatively stable if $\lambda h \leq 0$ (i.e., $\lambda \leq 0$). Then, the sixth order Runge-Kutta method with seven stages is absolutely stable in the interval $-3.17 \leq \lambda h \leq 0$.

Algorithm:

1) Input the initial data $x_0, y_0, z_0, h, \lambda, N$
2) For $i = 1 : N$ do steps 3-10
3) For $l = 1 : 7$ do steps 4-5
4) Evaluate $k_l(i)$ using equation (9)
5) Evaluate $m_l(i)$ using equation (9)
6) Evaluate

$$k(i) = \frac{1}{120}(11k_1(i) + 81k_3(i) + 81k_4(i) - 32k_5(i) - 32k_6(i) + 11k_7(i))$$

7) Evaluate

$$m(i) = \frac{1}{120}(11m_1(i) + 81m_3(i) + 81m_4(i) - 32m_5(i) - 32m_6(i) + 11m_7(i))$$

8) Evaluate $y_{i+1} = y_i + k(i)$ using equation (8)
9) Evaluate $z_{i+1} = z_i + m(i)$ using equation (8)
10) Calculate point wise absolute errors

IV. Numerical Examples

The sixth order Runge-Kutta with seven stages method is implemented on three model examples of second order initial value problems of Bratu-type equations and the results are compared with their exact solutions, and with the numerical solutions obtained by using Runge-Kutta method of order 5 by [9].

Example 4.1: Consider the Bratu-type initial value problem $y'' - 2e^y = 0,\ 0 < x < 1,\ y(0) = 0,\ y'(0) = 0$ whose exact solution is $y(x) = -2\ln(\cos x)$.

TABLE II
Comparison of absolute errors of Example 4.1 for $h = 0.1$.

x	Habtam et al. (RK5)	Current method (RK6)
0.1	$4.9027e^{-9}$	$4.7354e^{-11}$
0.2	$1.0071e^{-8}$	$4.4009e^{-10}$
0.3	$1.5973e^{-8}$	$1.2764e^{-9}$
0.4	$2.3233e^{-8}$	$2.7775e^{-9}$
0.5	$3.2789e^{-8}$	$5.3998e^{-9}$
0.6	$4.6204e^{-8}$	$1.0104e^{-8}$
0.7	$6.6335e^{-8}$	$1.9048e^{-8}$
0.8	$9.8959e^{-8}$	$3.7529e^{-8}$
0.9	$1.5718e^{-7}$	$8.0833e^{-8}$
1.0	$2.7544e^{-7}$	$1.9336e^{-7}$

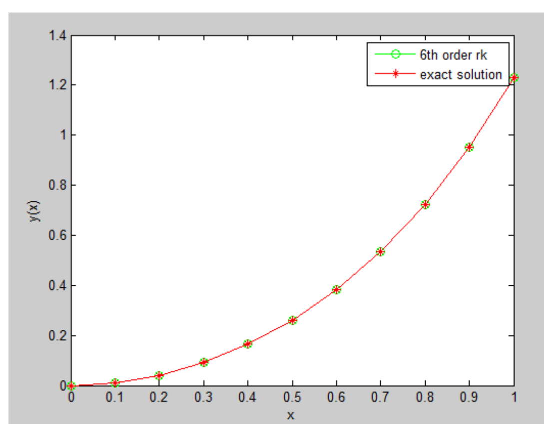

Fig. 1. Plot of exact and approximated solution of Example 4.1 for $h = 0.1$.

Example 4.2: Consider the Bratu-type initial value problem $y'' + \pi^2 e^{-y} = 0,\ 0 < x < 1,\ y(0) = 0,\ y'(0) = \pi$ whose exact solution is $y(x) = \ln(1 + \sin(\pi x))$.

Example 4.3: Consider the Bratu-type initial value problem $y'' - e^{2y} = 0,\ 0 < x < 1,\ y(0) = 0,\ y'(0) = 0$ whose exact solution is $y(x) = \ln(\sec x)$.

TABLE III
COMPARISON OF ABSOLUTE ERRORS OF EXAMPLE 4.1 FOR $h = 0.01$.

x	Habtam et al. (RK5)	Current method (RK6)
0.01	$4.8904e^{-15}$	$2.9300e^{-17}$
0.02	$9.7697e^{-15}$	$4.8247e^{-17}$
0.03	$1.4542e^{-14}$	$4.2718e^{-17}$
0.04	$1.9526e^{-14}$	$7.0256e^{-17}$
0.05	$2.4300e^{-14}$	$3.8164e^{-17}$
0.06	$2.9238e^{-14}$	$2.6021e^{-18}$
0.07	$3.4202e^{-14}$	$5.0307e^{-17}$
0.08	$3.9077e^{-14}$	$1.3878e^{-17}$
0.09	$4.4138e^{-14}$	$7.9797e^{-17}$
0.10	$4.9098e^{-14}$	$3.9899e^{-17}$

TABLE IV
COMPARISON OF ABSOLUTE ERRORS OF EXAMPLE 4.2 FOR $h = 0.1$.

x	Habtam et al. (RK5)	Current method (RK6)
0.1	$8.3507e^{-8}$	$3.7804e^{-7}$
0.2	$4.4458e^{-8}$	$5.4006e^{-7}$
0.3	$2.6380e^{-7}$	$6.5775e^{-7}$
0.4	$5.3872e^{-7}$	$7.7979e^{-7}$
0.5	$8.6014e^{-7}$	$9.2484e^{-7}$
0.6	$1.2299e^{-6}$	$1.1035e^{-6}$
0.7	$1.6559e^{-6}$	$1.3242e^{-6}$
0.8	$2.1494e^{-6}$	$1.5934e^{-6}$
0.9	$2.7194e^{-6}$	$1.9045e^{-6}$
1.0	$3.3454e^{-6}$	$2.1817e^{-6}$

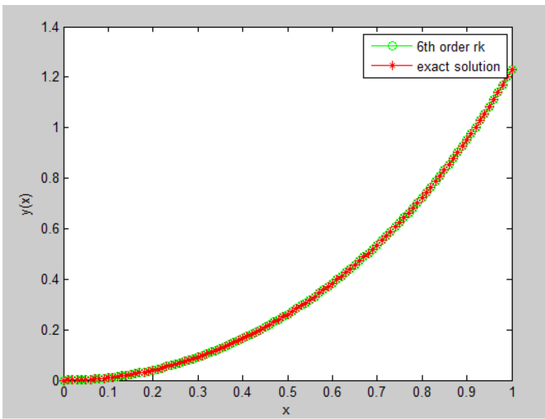

Fig. 2. Plot of exact and approximated solutions of Example 4.1 for $h = 0.01$.

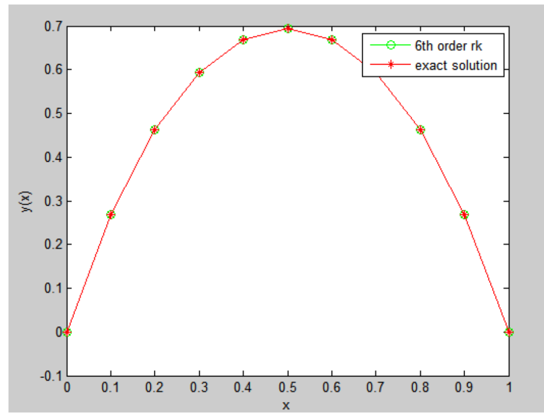

Fig. 3. Plot of exact and approximated solutions of Example 4.2 for $h = 0.1$.

V. DISCUSSIONS AND CONCLUSIONS

In this paper, higher order Runge-Kutta (RK6) method with seven stages is formulated to approximate the solutions of second order initial value problems of Bratu-type equation. The stability of the method is also discussed. To justify the applicability of the proposed method, tables of point-wise absolute errors and graphs have been plotted for three model examples to compare the exact solution and numerical solution at different mesh size h. From Tables II-VII, it is evident that all the absolute errors decrease rapidly as the mesh size h decreases which in turn show that smaller mesh size provides the better approximate value. Figures 1-6 show that the present method approximates the exact solution in an excellent manner.

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TABLE V
COMPARISON OF ABSOLUTE ERRORS OF EXAMPLE 4.2 FOR $h = 0.01$.

x	Habtam et al. (RK5)	Current method (RK6)
0.01	$2.2668e^{-13}$	$5.9442e^{-14}$
0.02	$4.1313e^{-13}$	$1.1242e^{-13}$
0.03	$5.6410e^{-13}$	$1.5964e^{-13}$
0.04	$6.8325e^{-13}$	$2.0214e^{-13}$
0.05	$7.7396e^{-13}$	$2.4061e^{-13}$
0.06	$8.3944e^{-13}$	$2.7536e^{-13}$
0.07	$8.8199e^{-13}$	$3.0720e^{-13}$
0.08	$9.0389e^{-13}$	$3.3654e^{-13}$
0.09	$9.0730e^{-13}$	$3.6360e^{-13}$
0.10	$8.9412e^{-13}$	$3.8858e^{-13}$

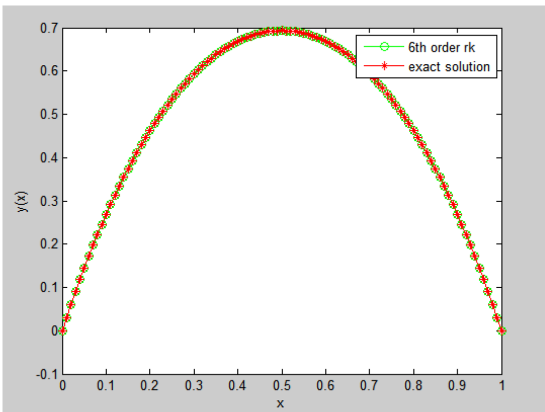

Fig. 4. Plot of exact and approximated solutions of Example 4.2 for $h = 0.01$.

TABLE VI
COMPARISON OF ABSOLUTE ERRORS FOR EXAMPLE 4.3 FOR $h = 0.1$.

x	Habtam et al. (RK5)	Current method (RK6)
0.1	$2.4514e^{-9}$	$2.3677e^{-11}$
0.2	$5.0353e^{-9}$	$2.2004e^{-10}$
0.3	$7.9863e^{-9}$	$6.3819e^{-10}$
0.4	$1.1616e^{-8}$	$1.3888e^{-9}$
0.5	$1.6395e^{-8}$	$2.6999e^{-9}$
0.6	$2.3102e^{-8}$	$5.0521e^{-9}$
0.7	$3.3168e^{-8}$	$9.5241e^{-9}$
0.8	$4.9480e^{-8}$	$1.8764e^{-8}$
0.9	$7.8591e^{-8}$	$4.0066e^{-8}$
1.0	$1.3772e^{-7}$	$9.6681e^{-8}$

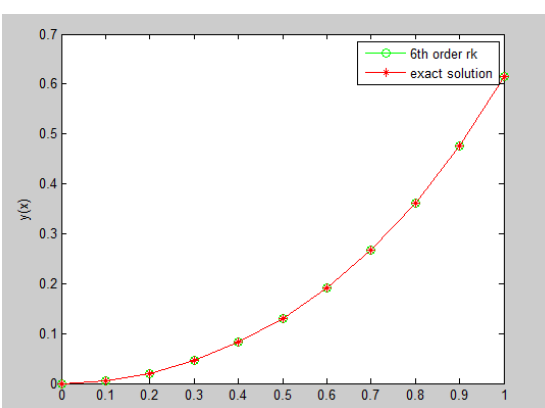

Fig. 5. Plot of exact and approximated solutions of Example 4.3 for $h = 0.1$.

TABLE VII
COMPARISON OF ABSOLUTE ERRORS OF EXAMPLE 4.3 FOR $h = 0.01$.

x	Habtam et al. (RK5)	Current method (RK6)
0.01	$2.3599e^{-15}$	$7.0582e^{-17}$
0.02	$4.8279e^{-15}$	$3.2878e^{-17}$
0.03	$7.3091e^{-15}$	$1.6697e^{-17}$
0.04	$9.6700e^{-15}$	$5.7896e^{-17}$
0.05	$1.2101e^{-14}$	$6.8522e^{-17}$
0.06	$1.4654e^{-14}$	$3.6429e^{-17}$
0.07	$1.7124e^{-14}$	$4.8138e^{-17}$
0.08	$1.9546e^{-14}$	$8.6736e^{-19}$
0.09	$2.2018e^{-14}$	$1.1276e^{-17}$
0.10	$2.4441e^{-14}$	$8.7604e^{-17}$

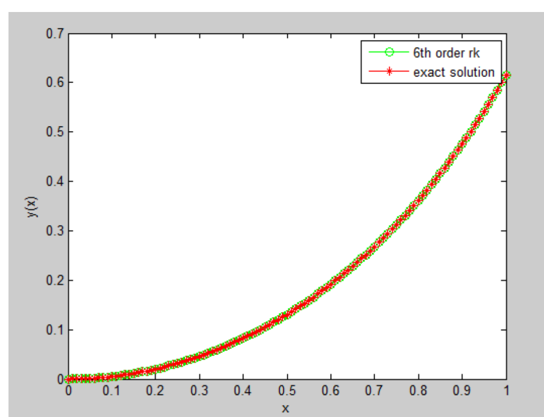

Fig. 6. Plot of exact and approximated solutions of Example 4.3 for $h = 0.01$.