I-Vague Vector Spaces

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***Abstract*—The notions of I-vague vector spaces of vector spaces with membership and non-membership functions taking values in an involutary dually residuated lattice ordered semigroup are introduced which generalizes the notions with truth values in a Boolean algebra as well as those usual vague sets whose membership and non-membership functions taking values in the unit interval [0, 1]. We discuss some properties of I-vague vector spaces.**

***Index Terms*—Involutary dually residuated lattice ordered semigroup, I-vague sets, I-vague vector spaces.**

I. Introduction

RAMAKRISHNA and Eswarlal [1] studied Boolean vague sets where the vague set of the universe X is defined by the pair of functions $(t_A,\ f_A)$ where t_A and f_A are mappings from a set X into a Boolean algebra satisfying the condition $t_A(x) \leq f_A(x)^{'}$ for all $x \in X$ where $f_A(x)^{'}$ is the complement of $f_A(x)$ in the Boolean algebra. K.L.N Swamy [2], [3], [4] introduced the concept of a Dually Residuated Lattice Ordered Semigroup (in short DRL-semigroup) which is a common abstraction of Boolean algebras and lattice ordered groups. The subclass of DRL-semigroups which are bounded and involutary (i.e. having 0 as the least, 1 as the greatest and satisfying $1-(1-x)=x$) which is categorically equivalent to the class of MV-algebras of Chang [5] and well studied offer a natural generalization of the closed unit interval [0, 1] of real numbers as well as Boolean algebras. Thus, the study of vague sets $(t_A,\ f_A)$ with values in an involutary DRL-semigroup promises a unified study of real valued vague sets and also those Boolean valued vague sets.

T. Eswarlal and N. Ramakrishna [6] studied vague fields and vector spaces. Moreover, K.V. Rama Rao and Amarendra Babu V. [7] studied vague vector spaces and vague Modules. In this paper, using the definition of I-vague sets in [8], we defined and studied I-vague vector spaces where I is an involutary DRL-semigroup which generalizes the work of vector spaces discussed in T. Eswarlal and N. Ramakrishna [6] and K.V. Rama Rao and Amarendra Babu V. [7].

II. Preliminaries

Definition 1: A system $A=(A,+,\leq,-)$ is called a dually residuated lattice ordered semigroup (in short DRL-semigroup) if and only if

i) $A=(A,\ +)$ is a commutative semigroup with zero "0";
ii) $A=(A,\ \leq)$ is a lattice such that $a+(b\cup c)=(a+b)\cup(a+c)$ and $a+(b\cap c)=(a+b)\cap(a+c)$ for all $a,\ b,\ c\in A$;
iii) Given $a,\ b\in A$, there exists a least x in A such that $b+x\geq a$, and we denote this x by $a-b$ (for a given $a,\ b$ this x is uniquely determined);
iv) $(a-b)\ \cup\ 0+b\leq a\cup b$ for all $a,\ b\in A$;
v) $a-a\geq 0$ for all $a\in A$.

Theorem 1: Any DRL-semigroup is a distributive lattice.

Definition 2: A DRL-semigroup A is said to be involutary if there is an element 1 ($\neq 0$) (0 is the identity w.r.t. +) such that

(i) $a+(1-a)=1+1$;
(ii) $1-(1-a)=a$ for all $a\in$ A.

Theorem 2: In a DRL-semigroup with 1, 1 is unique.

Theorem 3: If a DRL-semigroup contains a least element x, then $x=0$. Dually, if a DRL-semigroup with 1 contains a largest element α, then $\alpha=1$.

Throughout this paper let $I=(I,+,-,\vee,\wedge,0,1)$ be a dually residuated lattice ordered semigroup satisfying $1-(1-a)=a$ for all $a\in I$.

Lemma 1: Let 1 be the largest element of I. Then for $a,\ b\in$ I, the following holds

(i) $a+(1-a)=1$;
(ii) $1-a=1-b \Longleftrightarrow a=b$;
(iii) $1-(a\cup b)=(1-a)\cap(1-b)$.

Lemma 2: Let I be complete. If $a_\alpha\in I$ for every $\alpha\in\Delta$, then

(i) $1-\bigvee_{\alpha\in\Delta} a_\alpha=\bigwedge_{\alpha\in\Delta}(1-a_\alpha)$.
(ii) $1-\bigwedge_{\alpha\in\Delta} a_\alpha=\bigvee_{\alpha\in\Delta}(1-a_\alpha)$.

Definition 3: An I-vague set A of a non-empty set W is a pair $(t_A,\ f_A)$ where $t_A: W\to I$ and $f_A: W\to I$ with $t_A(x)\leq 1-f_A(x)$ for all $x\in W$.

Definition 4: The interval $[t_A(x),\ 1-f_A(x)]$ is called the I-vague value of $x\in$ W and is denoted by $V_A(x)$.

Definition 5: Let $B_1=[a_1,\ b_1]$ and $B_2=[a_2,\ b_2]$ be two I-vague values. We say $B_1\geq B_2$ if and only if $a_1\geq a_2$ and $b_1\geq b_2$.

Definition 6: Let $A=(t_A,\ f_A)$ and $B=(t_B,f_B)$ be I-vague sets on a non empty set W. A is said to be contained in B written as $A\subseteq B$ if and only if $t_A(x)\leq t_B(x)$ and $f_A(x)\geq f_B(x)$ for all $x\in$ W. A is said to be equal to B written as A = B if and only if $A\subseteq B$ and $B\subseteq A$.

Definition 7: An I-vague set A of W with $V_A(x)=V_A(y)$ for all $x,\ y\in W$ is called a constant I-vague set of W.

Definition 8: Let A be an I-vague set of a non empty set W. Let $A_{(\alpha,\beta)}=\{x\in W: V_A(x)\geq[\alpha,\beta]\}$ where $\alpha,\ \beta\in I$ and $\alpha\leq\beta$. Then $A_{(\alpha,\ \beta)}$ is called the $(\alpha,\ \beta)$ cut of the I-vague set A.

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Definition 9: Let S $\subseteq$W. The characteristic function of S denoted as $\chi_S = (t_{\chi_S},\ f_{\chi_S})$, which takes values in I is defined as follows:

$$t_{\chi_S}(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases} \quad f_{\chi_S}(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{otherwise.} \end{cases}$$

χ_S is called the I-vague characteristic set of S in I. Thus

$$V_{\chi_S}(x) = \begin{cases} [1,\ 1] & \text{if } x \in S \\ [0,\ 0] & \text{otherwise.} \end{cases}$$

Definition 10: Let $A = (t_A,\ f_A)$ and $B = (t_B,\ f_B)$ be I-vague sets of a set W.

(i) Their union $A \cup B$ is defined as $A \cup B = (t_{A\cup B},\ f_{A\cup B})$ where $t_{A\cup B}(x) = t_A(x) \vee t_B(x)$ and $f_{A\cup B}(x) = f_A(x) \wedge f_B(x)$ for each $x \in$ W.

(ii) Their intersection $A \cap B$ is defined as $A \cap B = (t_{A\cap B},\ f_{A\cap B})$ where $t_{A\cap B}(x) = t_A(x) \wedge t_B(x)$ and $f_{A\cap B}(x) = f_A(x) \vee f_B(x)$ for each $x \in$ G.

Definition 11: Let $B_1 = [a_1,\ b_1]$ and $B_2 = [a_2,\ b_2]$ be I-vague values. Then

(i) isup$\{B_1, B_2\} = [\sup\{a_1,\ a_2\},\ \sup\{b_1,\ b_2\}]$.

(ii) iinf$\{B_1, B_2\} = [\inf\{a_1,\ a_2\},\ \inf\{b_1,\ b_2\}]$.

Lemma 3: Let A and B be I-vague sets of a set W. Then $A \cup B$ and $A \cap B$ are also I-vague sets of W.

Let $x \in$ W. From the definition of $A \cup B$ and $A \cap B$ we have

(i) $V_{A\cup B}(x) =$ isup$\{V_A(x),\ V_B(x)\}$;

(ii) $V_{A\cap B}(x) =$iinf$\{V_A(x),\ V_B(x)\}$.

Definition 12: Let I be complete and $\{A_i = (t_{A_i},\ f_{A_i}) :\ i \in \triangle\}$ be a non empty family of I vague sets of W. Then for each $x \in W$,

(i) isup$\{V_{A_i}(x) :\ i \in \triangle\} = [\bigvee_{i\in\triangle} t_{A_i}(x),\ \ \bigvee_{i\in\triangle} (1 - f_{A_i})(x)]$.

(i) iinf$\{V_{A_i}(x) :\ i \in \triangle\} = [\bigwedge_{i\in\triangle} t_{A_i}(x),\ \ \bigwedge_{i\in\triangle} (1 - f_{A_i})(x)]$.

Lemma 4: Let I be complete. If $\{A_i : i \in \triangle\}$ is a non empty family of I-vague sets of W, then $\bigcup_{i\in\triangle} A_i$ and $\bigcap_{i\in\triangle} A_i$ are also an I-vague sets of W.

Definition 13: Let I be complete and $\{A_i = (t_{A_i},\ f_{A_i}) :\ i \in \triangle\}$ be a non empty family of I vague sets of W. Then for each $x \in W$,

(i) isup$\{V_{A_i}(x) :\ i \in \triangle\} = [\bigvee_{i\in\triangle} t_{A_i}(x),\ \ \bigvee_{i\in\triangle} (1 - f_{A_i})(x)]$.

(ii) iinf$\{V_{A_i}(x) :\ i \in \triangle\} = [\bigwedge_{i\in\triangle} t_{A_i}(x),\ \ \bigwedge_{i\in\triangle} (1 - f_{A_i})(x)]$.

Definition 14: Let $\Phi : X \to Y$ be a mapping from a set X into a set Y. Let B be an I-vague set of Y. Then the preimage of B, $\Phi^{-1}(B) = (t_{\Phi^{-1}(B)}, f_{\Phi^{-1}(B)})$ is given by $t_{\Phi^{-1}(B)} : X \to I$ and $f_{\Phi^{-1}(B)} : X \to I$ where $t_{\Phi^{-1}(B)}(x) = t_B(\Phi(x))$ and $f_{\Phi^{-1}(B)}(x) = t_B(\Phi(x))$ for each $x \in X$.

Lemma 5: Let $\Phi : X \to Y$ be a mapping from a set X into a set Y. If B be an I-vague set of Y, then $\Phi^{-1}(B)$ is an I-vague set of X and $V_{\Phi^{-1}(B)}(x) = V_B\Phi(x)$ for each $x \in X$.

Definition 15: Let I be complete and $\Phi : X \to Y$ be a mapping from a set X into a set Y. Let A = $(t_A,\ f_A)$ be an I-vague set of X. Then the image of A, $\Phi(A) = (t_{\Phi(A)}, f_{\Phi(A)})$ is given by

$$t_{\Phi(A)}(y) = \begin{cases} \bigvee_{x\in\Phi^{-1}(y)} t_A(z) & \text{if } \Phi^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\Phi(A)}(y) = \begin{cases} \bigwedge_{x\in\Phi^{-1}(y)} t_A(z) & \text{if } \Phi^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 6: Let I be complete and $\Phi : X \to Y$ be a mapping from a set X into a set Y. If A be an I-vague set of X, then $\Phi(A)$ is an I-vague set of Y.

Theorem 4: Let I be complete and $\Phi : X \to Y$ be a mapping from a set X into a set Y. If A is an I-vague set of X, then

$$V_{\Phi(A)}(y) = \begin{cases} isup\{V_A(z) : z \in \Phi^{-1}(y)\} & \text{if } \Phi^{-1}(y) \neq \emptyset \\ [0,0] & \text{otherwise.} \end{cases}$$

III. I-Vague Vector Spaces

Definition 16: Let W be a vector space over a field F and A be an I-vague set of W. Then A is said to be an I-vague subspace of W if

(i) $V_A(x+y) \geq iinf\{V_A(x), V_A(y)\}$

(ii) $V_A(\lambda x) \geq V_A(x)$ for all $x, y \in$W and $\lambda \in F$

Example 1: Let I be the unit interval $[0,1]$ of real numbers. Let $a \oplus b$ = min $\{1, a+b\}$. with the usual ordering $(I, \oplus, \leq, -)$ is an involutary DRL-semigroup. Consider the vector space $W = \Re^2$ over $\Re$. Let $A = (t_A,\ f_A)$ where $t_A : \Re^2 \to [0,1]$ by $t_A(x,y)$=1 and $f_A : \Re^2 \to [0,1]$ by $f_A(x,y)$=0 for all $(x,y) \in \Re^2$. Then A is an I-vague subspace of W.

Lemma 7: Let A be an I-vague subspace of W. Then

(i) $V_A(0) \geq V_A(x)$ for all $x \in$ W.

(ii) $V_A(\lambda x) = V_A(x)$ for all $x \in$ W and $\lambda \neq 0$.

Proof: Let A be an I-vague subspace of W.

(i) $V_A(0) \doteq V_A(0x) \geq V_A(x)$. Hence $V_A(0) \geq V_A(x)$ for all $x \in$W.

(ii) Let $\lambda \neq 0$ and $x \in W$. Then $V_A(x) = V_A((\lambda^{-1}\lambda)x) = V_A((\lambda^{-1})(\lambda x)) \geq V_A(\lambda x) \geq V_A(x)$.
Hence $V_A(\lambda x) = V_A(x)$ for all $\lambda \in F\backslash\{0\}$. ■

Lemma 8: Let W be a vector space over a field F. A is an I-vague subspace of W iff $V_A(\lambda x + \mu y) \geq iinf\{V_A(x), V_A(y)\}$ for all $\lambda, \mu \in F$ and $x, y \in W$.

Proof: Let A be an I-vague subspace of W. Let $x, y \in$ W and $\lambda, \mu \in$ F. Then $V_A(\lambda x) \geq V_A(x)$ and $V_A(\mu y) \geq V_A(y)$. Since A is an I-vague subspace of W, $V_A(\lambda x + \mu y) \geq iinf\{V_A(\lambda x), V_A(\mu y)\}$. Moreover iinf$\{V_A(\lambda x), V_A(\mu y)\} \geq iinf\{V_A(x), V_A(y)\}$. Hence $V_A(\lambda x + \mu y) \geq iinf\{V_A(x), V_A(y)\}$. Conversely, suppose that $V_A(\lambda x + \mu y) \geq iinf\{V_A(x), V_A(y)\}$ for all $\lambda, \mu \in F$ and $x, y \in W$. Put $\lambda = \mu = 1$. Then $V_A(x+y) \geq iinf\{V_A(x), V_A(y)\}$. Moreover,$V_A(\lambda x) = V_A(\lambda x + 0x) \geq iinf\{V_A(x), V_A(x)\} = V_A(x)$. This proves the lemma. ■

Moreover,$V_A(x-y) = V_A(x+-1y) \geq iinf\{V_A(x), V_A(y)\}$.

Lemma 9: Let W be a vector space over a field F and A be an I-vague subspace of W. Then $V_A(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \geq iinf\{V_A(x_1), V_A(x_2 + \ldots + V_A(x_n)\}$ for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$ and $x_1, x_2, \ldots, x_n \in W$.

Proof: We use proof by induction. Clearly the statement is true for $n = 2$. Assume that the statement is true for n.

$$V_A(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n + \lambda_{n+1} x_{n+1})$$
$$= V_A((\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) + \lambda_{n+1} x_{n+1})$$

$$\geq \text{iinf}\{V_A(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n), V_A(\lambda_{n+1} x_{n+1})\}$$
$$\geq \text{iinf}\{\text{iinf}\{V_A(x_1), V_A(x_2), ..., V_A(x_n)\}, V_A(x_{n+1})\}$$
$$= \text{iinf}\{V_A(x_1), V_A(x_2), ..., V_A(x_n), V_A(x_{n+1})\}$$

Therefore $V_A(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n + \lambda_{n+1} x_{n+1}) \geq iinf\{V_A(x_1), V_A(x_2), ..., V_A(x_n), V_A(x_{n+1})\}$. Hence the lemma follows. ■

Theorem 5: An I-vague set A of a vector space W is an I-vague subspace of W iff for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha,\beta)}$ of W whenever it is non empty.

Proof: Let A be an I-vague set of a vector space W. Suppose that A is an I-vague subspace of W. We prove that $A_{(\alpha,\beta)}$ is a subspace of W whenever it is non empty. Let $x, y \in A_{(\alpha,\beta)}$. Then $V_A(x) \geq [\alpha, \beta]$ and $V_A(y) \geq [\alpha, \beta]$. It follows that $\text{iinf}\{V_A(x), V_A(y)\} \geq [\alpha, \beta]$. Since $V_A(x+y) \geq iinf\{V_A(x), V_A(y)\}, V_A(x+y) \geq [\alpha, \beta]$. Hence $x+y \in A_{(\alpha,\beta)}$. Let $x \in A_{(\alpha,\beta)}$ and $\lambda \in F$. Then $V_A(\lambda x) \geq V_A(x) \geq [\alpha, \beta]$. Hence $\lambda x \in A_{(\alpha,\beta)}$. Therefore $A_{(\alpha,\beta)}$ is a subspace of W.

Conversely, suppose that $A_{(\alpha,\beta)}$ is a subspace of W whenever it is non empty. We prove that A is an I-vague subspace of W. Let $x, y \in W$. Suppose $V_A(x) = [\alpha, \beta]$ and $V_A(y) = [\gamma, \delta]$ for some $\alpha, \beta, \gamma, \delta \in I$. $\text{iinf}\{V_A(x), V_A(y)\} = [\alpha \wedge \gamma, \beta \wedge \delta] = [\xi, \eta]$ for some $\xi, \eta \in I$. Hence $x, y \in A_{(\xi,\eta)}$. Since $A_{(\xi,\eta)}$ is a subspace of W, $\lambda x + \mu y \in A_{(\xi,\eta)}$ for $\lambda, \mu \in F$. Hence $V_A(\lambda x + \mu y) \geq [\xi, \eta] = \text{iinf}\{V_A(x), V_A(y)\}$. Thus, $V_A(\lambda x + \mu y) \geq \text{iinf}\{V_A(x), V_A(y)\}$. Hence the theorem follows. ■

Lemma 10: Let A be an I-vague subspace of a vector space W. Then the set $W_A = \{x \in W : V_A(x) = V_A(0)\}$ is a subspace of W.

Proof: Since $0 \in W_A$, $W_A \neq \emptyset$. Let $x, y \in W_A$. Then $V_A(x) = V_A(y) = V_A(0)$. Hence $V_A(x+y) \geq \text{iinf}\ \{V_A(x), V_A(y)\} = V_A(0)$. Since $V_A(0) \geq V_A(x+y)$,$V_A(x+y) = V_A(0)$. Hence $x+y \in W_A$. Let $\lambda \in$ F and $x \in W_A$. Then $V_A(x) = V_A(0)$. $V_A(\lambda x) \geq V_A(x) = V_A(0)$. Thus $V_A(\lambda x) = V_A(0)$. Hence $\lambda x \in W_A$. Therefore W_A is a subspace of W. ■

Lemma 11: Let U be a subspace of a vector space W with $\alpha, \beta, \gamma, \delta \in I$, $\alpha \leq \beta, \gamma \leq \delta$ and $[\gamma,\ \delta] \leq [\alpha,\ \beta]$. Then the I-vague set A of W defined by

$$V_A(x) = \begin{cases} [\alpha,\ \beta] & \text{if } x \in U \\ [\gamma,\ \delta] & \text{otherwise.} \end{cases}$$

is an I-vague subspace of W.

Proof: Let U be a subspace of W. We have the following three cases:

(i) Let $x, y \in U$. Since U is a subspace of W, $\lambda x + \mu y \in U$ for $\lambda, \mu \in F$. $V_A(\lambda x + \mu y) = [\alpha,\ \beta] = \text{iinf}\ \{V_A(x), V_A(y)\}$. It follows that $V_A(\lambda x + \mu y) \geq \text{iinf}\ \{V_A(x), V_A(y)\}$.

(ii) Exactly one of x or y does not belong to U. Suppose $x \in U$ and $y \notin U$. $\lambda x + \mu y \notin U$ for any $\mu \neq 0$. $V_A(\lambda x + \mu y) = [\gamma,\ \delta]$=iinf $\{V_A(x), V_A(y)\}$. Hence $V_A(\lambda x + \mu y) \geq \text{iinf}\ \{V_A(x), V_A(y)\}$.

(iii) Both x and y does not belong to U. $\lambda x + \mu y \notin U$ for any $\lambda, \mu, \neq 0$. $\text{iinf}\ \{V_A(x), V_A(y)\} = [\gamma,\ \delta] = V_A(\lambda x + \mu y)$. Hence $V_A(\lambda x + \mu y) \geq \text{iinf}\ \{V_A(x), V_A(y)\}$.

This proves the lemma. ■

Lemma 12: Let A and B be I-vague subspaces of a vector space W. Then $A \cap B$ is also I-vague subspace of W.

Proof: Let A and B be I-vague subspaces of W. We prove that $A \cap B$ is also an I-vague subspace of W. By Lemma 3,$A \cap B$ is an I-vague set of W. Let $x, y \in$ W.

$$\begin{aligned} V_{A\cap B}(x+y) &= \text{iinf}\{V_A(x+y), V_B(x+y)\} \\ &\geq \text{iinf}\{iinf\{V_A(x), V_A(y)\}, iinf\{V_B(x), V_B(y)\}\} \\ &= \text{iinf}\{iinf\{\{V_A(x), V_B(x)\}, iinf\{\{V_A(y), V_B(y)\}\} \\ &= \text{iinf}\{V_{A\cap B}(x), V_{A\cap B}(y)\} \end{aligned}$$

Hence $V_{A\cap B}(x+y) \geq \text{iinf}\{V_{A\cap B}(x), V_{A\cap B}(y)\}$. $V_{A\cap B}(\lambda x) = \text{iinf}\ \{V_A(\lambda x), V_B(\lambda x)\} \geq \text{iinf}\{V_A(x), V_B(x)\} = V_{A\cap B}(x)$. Thus $V_{A\cap B}(\lambda x) \geq V_{A\cap B}(x)$. Therefore $A \cap B$ is an I-vague subspace of W. ■

Lemma 13: Let I be complete. If $\{A_i : i \in \triangle\}$ is a non empty family of I-vague subspaces of W, then $\bigcap_{i\in\triangle} A_i$ is an I-vague subspace of W.

Proof: Let $\{A_i : i \in \triangle\}$ be a non empty family of I-vague subspaces of W. Let $A = \bigcap_{i\in\triangle} A_i$. We prove that A is an I-vague subspace of W. By Lemma 4, A is an I-vague set of W.Let $x, y \in W$. Then

$$\begin{aligned} V_A(x+y) &= \text{iinf}\{V_{A_i}(x+y) : i \in \triangle\} \\ &\geq \text{iinf}\{\text{iinf}\{V_{A_i}(x), V_{A_i}(y)\} : i \in \triangle\} \\ &= \text{iinf}\{iinf\{V_{A_i}(x) : i \in \triangle\}, iinf\{V_{A_i}(y) : i \in \triangle\}\} \\ &= \text{iinf}\{V_A(x), V_A(y)\}. \end{aligned}$$

Thus $V_A(x+y) \geq \text{iinf}\{V_A(x), V_A(y)\}$. $V_A(\lambda x) = \text{iinf}\{V_{A_i}(\lambda x) : i \in \triangle\}\} \geq \text{iinf}\{V_{A_i}(x) : i \in \triangle\}\}$ $=V_A(x)$. Hence the lemma follows. ■

Example 2: Consider $W = \Re^2$ over $\Re$.Then $W_1 = \{(x, y) : x + y = 0\}$
and $W_2 = \{(x, y) : x - y = 0\}$ are subspaces of W.

$$V_A(x) = \begin{cases} [\alpha,\ \beta] & \text{if } x \in W_1 \\ [\gamma,\ \delta] & \text{otherwise.} \end{cases} \qquad V_B(x) = \begin{cases} [\alpha,\ \beta] & \text{if } x \in W_2 \\ [\gamma,\ \delta] & \text{otherwise.} \end{cases}$$

with $\alpha, \beta, \gamma, \delta \in I$, $\alpha \leq \beta, \gamma \leq \delta$ and $[\alpha,\ \beta] \leq [\gamma,\ \delta]$. We show that $A \cup B$ is not an I-vague subspace of W. Let u =(1,-1) and v=(1,1).

$$V_{A\cup B}(u+v) = V_{A\cup B}(2,0) = \text{isup}\{V_A(2,0), V_B(2,0) = [\gamma,\ \delta].$$
$$V_{A\cup B}(u) = V_{A\cup B}(1,-1) = \text{isup}\{V_A(1,-1), V_B(1,-1) = [\alpha,\ \beta].$$
$$V_{A\cup B}(v) = V_{A\cup B}(1,1) = \text{isup}\{V_A(1,1), V_B(1,1)\} = [\alpha,\ \beta].$$
$$\text{iinf}\{V_{A\cup B}(u), V_{A\cup B}(v)\} = [\alpha,\ \beta].$$
$$V_{A\cup B}(u+v) = [\gamma,\ \delta] \ngeq [\alpha,\ \beta] = \text{iinf}\{V_{A\cup B}(u), V_{A\cup B}(v)\}.$$

Therefore $A \cup B$ is not an I-vague subspace of W.

Lemma 14: Let $U \neq \emptyset$. The I-vague characteristic function set of U, χ_U is an I-vague subspace of W iff U is a subspace of W.

Proof: Suppose that χ_U is an I-vague subspace of W. Let $x, y \in$ U. Then $V_{\chi_U}(x) = [1,\ 1]$ and $V_{\chi_U}(y) = [1, 1]$. Since χ_U is an I-vague subspace of W, $V_{\chi_U}(x+y) \geq \text{iinf}\{V_{\chi_U}(x), V_{\chi_U}(y)\} = [1,\ 1]$. Hence $V_{\chi_U}(x+y) = [1,\ 1]$. So, $x+y \in U$. $V_{\chi_U}(\lambda x) \geq V_{\chi_U}(x) = [1,\ 1]$. It follows that $V_{\chi_U}(\lambda x) = [1,\ 1]$. Hence $\lambda x \in U$. Therefore U is a subspace of W. Conversely, suppose that U is a subspace of W. Then Consider

$$V_{\chi_U}(x) = \begin{cases} [1,\ 1] & \text{if } x \in U \\ [0,\ 0] & \text{otherwise.} \end{cases}$$

By Lemma 11, χ_U is an I-vague subspace of W. ∎

Theorem 6: Let A be an I-vague subspace of a vector space W. If $V_A(x-y) = V_A(0)$ for all $x, y \in W$, then $V_A(x) = V_A(y)$.

Proof: Let A be an I-vague subspace of a vector space W. Suppose that $V_A(x-y) = V_A(0)$ for $x, y \in W$. We prove that $V_A(x) = V_A(y)$. $V_A(x-y) = V_A(0)$ implies that $V_A(y-x) = V_A(0)$.

$$\begin{aligned} V_A(x) &= V_A((x-y)+y). \\ &\geq \text{iinf}\{V_A(x-y), V_A(y)\} \\ &= \text{iinf}\{V_A(0), V_A(y)\} \\ &= V_A(y) \end{aligned}$$

Thus $V_A(x) \geq V_A(y)$. Similarly, $V_A(y) = V_A((y-x)+x) \geq \text{iinf}\{V_A(y-x), V_A(x)\} = \text{iinf}\{V_A(0), V_A(x)\} = V_A(x)$. Thus $V_A(y) \geq V_A(x)$. Hence $V_A(x) = V_A(y)$. ∎

The following example shows that the converse of the above theorem is not true.

Example 3: Let I be the unit interval $[0,\ 1]$ of real numbers. Define $a \oplus b = \min\{1, a+b\}$. With the usual ordering $(I, \oplus, \leq, -)$ is an involutary DRL-semigroup. Let $W = \Re^2$ over $\Re$. Then $U = \{(x,y) : x+2y = 0\}$ is a subspace of W. Define the I-vague subspace A of W by

$$V_A(u) = \begin{cases} [\frac{1}{2},\ 1] & \text{if } u \in U \\ [0,\ \frac{1}{4}] & \text{otherwise.} \end{cases}$$

Let $u = (-2, 2)$ and $v = (1, 2)$. $V_A(u) = V_A(v)$=$[0,\ \frac{1}{4}]$ and $V_A(u-v) = V_A(-3, 0)$=$[0,\ \frac{1}{4}] \neq V_A(0)$. Thus $V_A(u) = V_A(v)$ but $V_A(u-v) \neq V_A(0)$.

Theorem 7: Let A be an I-vague subspace of a vector space W and $x \in W$. Then $V_A(x+y) = V_A(y)$ for all $y \in W$ iff $V_A(x) = V_A(0)$.

Proof: Let A be an I-vague subspace of a vector space W and $x \in W$. Suppose that $V_A(x+y) = V_A(y)$ for all $y \in W$. Take $y = 0$. Hence $V_A(x) = V_A(0)$. Conversely, suppose that $V_A(x) = V_A(0)$. Let $y \in W$. Then $V_A(x+y) \geq \text{iinf}\{V_A(x), V_A(y)\} = V_A(y)$. It follows that $V_A(x+y) \geq V_A(y)$.

$$\begin{aligned} V_A(y) &= V_A(-x+x+y) \\ &\geq \text{iinf}\{V_A(-x), V_A(x+y)\} \\ &= \text{iinf}\{V_A(x), V_A(x+y)\} \\ &= \text{iinf}\{V_A(0), V_A(x+y)\} \\ &= V_A(x+y) \end{aligned}$$

Thus $V_A(y) \geq V_A(x+y)$. It follows that $V_A(x+y) = V_A(y)$. ∎

Theorem 8: Let A be an I-vague subspace of a vector space W. If $V_A(x-y) = V_A(0)$ for all $x, y \in W$, then $V_A(x) = V_A(y)$.

Proof: Let A be an I-vague subspace of a vector space W. $V_A(x) = V_A((x-y)+y) \geq \text{iinf}\{V_A(x-y), V_A(y)\} = \text{iinf}\{V_A(0), V_A(y)\} = V_A(y)$. Similarly, $V_A(y) = V_A((y-x)+x) \geq \text{iinf}\{V_A(y-x), V_A(x)\} = \text{iinf}\{V_A(0), V_A(x)\} = V_A(x)$. Hence $V_A(x) = V_A(y)$. ∎

Theorem 9: Let W_1 and W_2 be vector spaces over a field F, and let T be a linear transformation from W_1 into W_2. If A is an I-vague subspace of W_2, then $T^{-1}(A)$ is an I-vague subspace of W_1.

Proof: Let T be a linear transformation from W_1 into W_2 and A be an I-vague subspace of W_2.

$$\begin{aligned} V_{T^{-1}(A)}(\lambda x + \mu y) &= V_A(T(\lambda x + \mu y)) \\ &= V_A(\lambda T(x) + \mu T(y)) \\ &\geq \text{iinf}\{V_A(\lambda T(x)), V_A(\mu T(y))\} \\ &\geq \text{iinf}\{V_A(T(x)), V_A(T(y))\} \\ &= \text{iinf}\{V_{T^{-1}(A)}(x), V_{T^{-1}(A)}(y)\} \end{aligned}$$

This completes the proof. ∎

Theorem 10: Let I be complete and infinitely meet distributive. Let U and V be vector spaces over a field F and $T : U \to V$ be a linear transformation. If A is an I-vague subspace of U, then $T(A)$ is an I-vague subspace of V.

Proof: Let $T : U \to V$ be a linear transformation and A be an I-vague subspace of U.

$$\begin{aligned} V_{T(A)}(y_1 + y_2) &= \text{isup}\{V_A(z) : z \in T^{-1}(y_1 + y_2)\} \\ &\geq \text{isup}\{V_A(z) : z = x_1 + x_2 \text{ where } x_1 \in T^{-1}(y_1) \\ &\qquad \text{and } x_2 \in T^{-1}(y_2)\} \\ &= \text{isup}\{V_A(x_1 + x_2) : x_1 \in T^{-1}(y_1) \text{ and} \\ &\qquad x_2 \in T^{-1}(y_2)\} \\ &\geq \text{isup}\{\text{iinf}\{V_A(x_1), V_A(x_2)\} : x_1 \in T^{-1}(y_1) \\ &\qquad \text{and } x_2 \in T^{-1}(y_2)\} \\ &= \text{iinf}\{\text{isup}\{V_A(x_1), V_A(x_2)\} : x_1 \in T^{-1}(y_1) \\ &\qquad \text{and } x_2 \in T^{-1}(y_2)\} \\ &\text{since } I \text{ is infinitely meet distributive} \\ &= \text{iinf}\{V_{T(A)}(y_1), V_{T(A)}(y_2)\} \end{aligned}$$

$$\begin{aligned} V_{T(A)}(y) &= isup\{V_A(z) : z \in T^{-1}(y)\} \\ &= \text{isup}\{V_A(z) : T(z) = (y)\} \\ &\leq \text{isup}\{V_A(\lambda z) : T(z) = y \text{ for any } \lambda \in F\} \\ &= \text{isup}\{V_A(\lambda z) : T(\lambda z) = \lambda y\} \\ &= \text{isup}\{V_A(u) : T(u) = \lambda y\} \\ &= V_{T(A)}(\lambda y) \end{aligned}$$

This proves the theorem. ∎

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