I-Vague Vector Spaces
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Abstract—The notions of I-vague vector spaces of vector spaces with membership and non-membership functions taking values in an involutary dually residuated lattice ordered semigroup are introduced which generalizes the notions with truth values in a Boolean algebra as well as those usual vague sets whose membership and non-membership functions taking values in the unit interval [0, 1]. We discuss some properties of I-vague vector spaces.

Index Terms—Involutary dually residuated lattice ordered semigroup, I-vague sets, I-vague vector spaces.

I. INTRODUCTION

RAMA KRISHNA and Eswaral [1] studied Boolean vague sets where the vague set of the universe X is defined by the pair of functions \((f_t, f_A)\) where \(t\) and \(f_A\) are mappings from a set X into a Boolean algebra satisfying the condition \(t_A(x) \leq f_A(x)^t\) for all \(x \in X\) where \(f_A(x)^t\) is the complement of \(f_A(x)\) in the Boolean algebra. K.L.N Swamy [2], [3], [4] introduced the concept of a Dually Residuated Lattice Ordered Semigroup (in short DRL-semigroup) which is a common abstraction of Boolean algebra and lattice ordered groups. The subclass of DRL-semigroups which are bounded and involutary (i.e. having 0 as the least, 1 as the greatest and satisfying \(1 - (1 - x) = x\)) which is categorically equivalent to the class of MV-algebras of Chang [5] and well studied offer a natural generalization of the closed unit interval \([0, 1]\) of real numbers as well as Boolean algebras. Thus, the study of vague sets \((t_A, f_A)\) with values in an involutary DRL-semigroup promises a unified study of real valued vague sets and also those Boolean valued vague sets.

T. Eswaral and N. Ramakrishna [6] studied vague fields and vector spaces. Moreover, K.V. Rama Rao and Amarendra Babu V. [7] studied vague vector spaces and vague Modules. In this paper, using the definition of I-vague sets in [8], we defined and studied I-vague vector spaces where I is an involutary DRL-semigroup which generalizes the work of vector spaces discussed in T. Eswaral and N. Ramakrishna [6] and K.V. Rama Rao and Amarendra Babu V. [7].

II. PRELIMINARIES

Definition 1: A system \(A = (A, +, \leq, -)\) is called a dually residuated lattice ordered semigroup (in short DRL-semigroup) if and only if
i) \(A = (A, +)\) is a commutative semigroup with zero “0”;
ii) \(A = (A, \leq)\) is a lattice such that \(a + (b \cup c) = (a + b) \cup (a + c)\) and \(a + (b \cap c) = (a + b) \cap (a + c)\) for all \(a, b, c \in A\);

Definition 2: A DRL-semigroup \(A\) is said to be involutary if there is an element \(1 \neq 0\) (0 is the identity w.r.t. +) such that
(i) \(a + (1 - a) = 1 + 1\);
(ii) \(1 - (1 - a) = a\) for all \(a \in A\).

Theorem 2: In a DRL-semigroup with 1, 1 is unique.

Theorem 3: If a DRL-semigroup contains a least element \(x\), then \(x = 0\) is the identity w.r.t. +.

Throughout this paper let \(I = (I, +, -, \vee, \wedge, 0, 1)\) be a dually residuated lattice ordered semigroup satisfying \(1 - (1 - a) = a\) for all \(a \in I\).

Lemma 1: Let \(I\) be the largest element of \(I\). Then for \(a, b \in I\), the following holds
(i) \(a + (1 - a) = 1\);
(ii) \(1 - a = 1 - b \iff a = b\);
(iii) \(1 - (a \cup b) = (1 - a) \cap (1 - b)\).

Definition 3: An I-vague set \(A\) of a non-empty set \(W\) is a pair \((t_A, f_A)\) where \(t_A : W \rightarrow I\) and \(f_A : W \rightarrow I\) with \(t_A(x) \leq 1 - f_A(x)\) for all \(x \in W\).

Definition 4: The interval \([t_A(x), 1 - f_A(x)]\) is called the I-vague value of \(x \in W\) and is denoted by \(V_A(x)\).

Definition 5: Let \(B_1 = [a_1, b_1]\) and \(B_2 = [a_2, b_2]\) be two I-vague values. We say \(B_1 \geq B_2\) if and only if \(a_1 \geq a_2\) and \(b_1 \geq b_2\).

Definition 6: Let \(A = (t_A, f_A)\) and \(B = (t_B, f_B)\) be I-vague sets on a non-empty set \(W\). A is said to be contained in \(B\) written as \(A \subseteq B\) if and only if \(t_A(x) \leq t_B(x)\) and \(f_A(x) \geq f_B(x)\) for all \(x \in W\). A is said to be equal to \(B\) written as \(A = B\) if and only if \(A \subseteq B\) and \(B \subseteq A\).

Definition 7: An I-vague set \(A\) of \(W\) with \(V_A(x) = V_A(y)\) for all \(x, y \in W\) is called a constant I-vague set of \(W\).

Definition 8: Let \(A\) be an I-vague set of a non-empty set \(W\). Let \(A_{\alpha, \beta} = \{x \in W : V_A(x) \geq [\alpha, \beta]\}\) where \(\alpha, \beta \in I\) and \(\alpha \leq \beta\). Then \(A_{\alpha, \beta}\) is called the \((\alpha, \beta)\) cut of the I-vague set \(A\).
Definition 9: Let $S \subseteq W$. The characteristic function of $S$ denoted as $\chi_S(t_{\mathcal{X}_S}), f_{\mathcal{X}_S})$, which takes values in $I$ is defined as follows:

$$
t_{\mathcal{X}_S}(x) = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{otherwise}
\end{cases} \quad f_{\mathcal{X}_S}(x) = \begin{cases} 
0 & \text{if } x \in S \\
1 & \text{otherwise}.
\end{cases}
$$

$\chi_S$ is called the I-vague characteristic set of $S$ in $I$. Thus $V_{\mathcal{X}_S}(x) = \begin{cases} 
[1, 1] & \text{if } x \in S \\
[0, 0] & \text{otherwise}.
\end{cases}$

Definition 10: Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be I-vague sets of a set $W$.

(i) Their union $A \cup B$ is defined as $A \cup B = (t_{A \cup B}, f_{A \cup B})$ where $t_{A \cup B}(x) = t_A(x) \vee t_B(x)$ and $f_{A \cup B}(x) = f_A(x) \wedge f_B(x)$ for each $x \in W$.

(ii) Their intersection $A \cap B$ is defined as $A \cap B = (t_{A \cap B}, f_{A \cap B})$ where $t_{A \cap B}(x) = t_A(x) \wedge t_B(x)$ and $f_{A \cap B}(x) = f_A(x) \vee f_B(x)$ for each $x \in A \cap B$.

Definition 11: Let $B_1 = [a_1, b_1]$ and $B_2 = [a_2, b_2]$ be I-vague values. Then

(i) $\sup\{B_1, B_2\} = \sup\{a_1, a_2, b_1, b_2\}$.

(ii) $\inf\{B_1, B_2\} = \inf\{a_1, a_2, b_1, b_2\}$.

Lemma 4: Let $A \subseteq W$ be complete and $\{A_i : i \in \Delta\}$ be a non empty family of I-vague sets of $W$. Then for each $x \in W$,

(i) $\sup\{A_i : i \in \Delta\} = [\bigvee_{i \in \Delta} t_{A_i}(x), \bigvee_{i \in \Delta} (1 - f_{A_i}(x))].$

(ii) $\inf\{A_i : i \in \Delta\} = [\bigwedge_{i \in \Delta} t_{A_i}(x), \bigwedge_{i \in \Delta} (1 - f_{A_i}(x))].$

Definition 14: Let $\Phi : X \to Y$ be a mapping from a set $X$ into a set $Y$. Let $B$ be an I-vague set of $Y$. Then the preimage of $B$ under $\Phi$, $\Phi^{-1}(B)$ is given by $t_{\Phi^{-1}(B)}(x) = t_B(\Phi(x))$ and $f_{\Phi^{-1}(B)}(x) = f_B(\Phi(x))$ for each $x \in X$.

Lemma 5: Let $\Phi : X \to Y$ be a mapping from a set $X$ into a set $Y$. If $B$ be an I-vague set of $Y$, then $\Phi^{-1}(B)$ is an I-vague set of $X$ and $V_{\Phi^{-1}(B)}(x) = V_B(\Phi(x))$ for each $x \in X$.

Definition 15: Let $I$ be complete and $\Phi : X \to Y$ be a mapping from a set $X$ into a set $Y$. Let $A = (t_A, f_A)$ be an I-vague set of $X$. Then the image of $A$, $\Phi(A) = (t_{\Phi(A)}, f_{\Phi(A)})$ is given by

$$
t_{\Phi(A)}(y) = \begin{cases} 
\bigvee_{x \in \Phi^{-1}(y)} t_A(z) & \text{if } \Phi^{-1}(y) \neq \emptyset \\
0 & \text{otherwise}
\end{cases} \quad f_{\Phi(A)}(y) = \begin{cases} 
1 & \text{if } \Phi^{-1}(y) \neq \emptyset \\
0 & \text{otherwise}.
\end{cases}
$$

Lemma 6: Let $I$ be complete and $\Phi : X \to Y$ be a mapping from a set $X$ into a set $Y$. If $A$ be an I-vague set of $X$, then $\Phi(A)$ is an I-vague set of $Y$.

Theorem 4: Let $I$ be complete and $\Phi : X \to Y$ be a mapping from a set $X$ into a set $Y$. If $A$ be an I-vague set of $X$, then

$$
V_{\Phi(A)}(y) = \begin{cases} 
isup\{V_A(z) : z \in \Phi^{-1}(y)\} & \text{if } \Phi^{-1}(y) \neq \emptyset \\
[0, 0] & \text{otherwise}.
\end{cases}
$$

III. I-Vague Vector Spaces

Definition 16: Let $W$ be a vector space over a field $F$ and $A$ be an I-vague set of $W$. Then $A$ is said to be an I-vague subspace of $W$ if

(i) $v_a(x + y) \geq \inf\{V_A(x), V_A(y)\}$

(ii) $V_A(\lambda x) \geq V_A(x)$ for all $x, y \in W$ and $\lambda \in F$

Example 1: Let $I$ be the unit interval $[0, 1]$ of real numbers. Let $a \in b = \min\{1, a + b\}$, with the usual ordering $(I, \leq, \vee)$ is an involutory DRL-semigroup. Consider the vector space $W = R^2$ over $R$. Let $A = (t_A, f_A)$ where $t_A : R^2 \to [0, 1]$ by $t_A(x, y) = 1$ and $f_A : R^2 \to [0, 1]$ by $f_A(x, y) = 0$ for all $(x, y) \in R^2$. Then $A$ is an I-vague subspace of $W$.

Lemma 7: Let $A$ be an I-vague subspace of $W$. Then

(i) $V_A(0) \geq V_A(x)$ for all $x \in W$ and $\lambda \neq 0$

Proof: Let $A$ be an I-vague subspace of $W$.

(ii) $V_A(\lambda x) = V_A(x)$ for all $x \in W$ and $\lambda \neq 0$

Proof: Let $A$ be an I-vague subspace of $W$.

Lemma 8: Let $W$ be a vector space over a field $F$. $A$ is an I-vague subspace of $W$ iff $V_A(\lambda x + \mu y) \geq \inf\{V_A(x), V_A(y)\}$ for all $\lambda, \mu \in F$ and $x, y \in W$.

Proof: Let $A$ be an I-vague subspace of $W$.

Theorem 9: Let $W$ be a vector space over a field $F$ and $A$ be an I-vague subspace of $W$. Then $V_A(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \geq \inf\{V_A(x_1), V_A(x_2) + \ldots + V_A(x_n)\}$ for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$ and $x_1, x_2, \ldots, x_n \in W$.

Proof: We use proof by induction. Clearly the statement is true for $n = 2$. Assume that the statement is true for $n$. Then

$$
V_A(\lambda_{n+1} x_{n+1}) \geq \inf\{V_A(\lambda_{n+1} x_{n+1})\}.
$$
\[ \geq \inf \{ \{ \alpha \lambda_1 x_1 + \alpha \lambda_2 x_2 + \ldots + \lambda_n x_n \}, V_4(\alpha \lambda_1 x_1 + \alpha \lambda_2 x_2 + \ldots + \lambda_n x_n) \} \]

\[ \geq \inf \{ \inf \{ \{ \alpha \lambda_1 x_1 + \alpha \lambda_2 x_2 + \ldots + \lambda_n x_n \}, V_4(\alpha \lambda_1 x_1 + \alpha \lambda_2 x_2 + \ldots + \lambda_n x_n) \} \} \]

\[ = \inf \{ \{ \alpha \lambda_1 x_1 + \alpha \lambda_2 x_2 + \ldots + \lambda_n x_n \}, V_4(\alpha \lambda_1 x_1 + \alpha \lambda_2 x_2 + \ldots + \lambda_n x_n) \} \]

Therefore \( V_4(\alpha \lambda_1 x_1 + \alpha \lambda_2 x_2 + \ldots + \lambda_n x_n + \alpha \lambda_{n+1} x_{n+1}) \) is a subspace of \( W \) whenever it is non empty. \( \lambda \alpha \{ \) is a subspace of \( W \) whenever it is non empty. We prove that \( \lambda \alpha \) is an \( I \)-vague subspace of \( W \). Let \( x, y \in W \). Then \( \lambda \alpha(x) \geq \{ \alpha \beta \} \) and \( \lambda \alpha(y) \geq \{ \alpha \beta \} \).

Theorem 5: An \( I \)-vague set \( A \) of a vector space \( W \) is an \( I \)-vague subspace of \( W \) if for all \( \alpha, \beta \in I \) with \( \alpha \leq \beta \), the \( I \)-vague cut \( A(\alpha, \beta) \) of \( W \) whenever it is non empty.

Proof: Let \( A \) be an \( I \)-vague set of a vector space \( W \). Suppose that \( A \) is an \( I \)-vague subspace of \( W \). We prove that \( A(\alpha, \beta) \) is a subspace of \( W \) whenever it is non empty. Let \( x, y \in A(\alpha, \beta) \). Then \( \lambda \alpha(x) \geq \{ \alpha \beta \} \) and \( \lambda \alpha(y) \geq \{ \alpha \beta \} \).

Thus \( \lambda \alpha(x) + y \geq \{ \alpha \beta \} \). Since \( \lambda \alpha(x) + y \geq \inf \{ \{ \lambda \alpha(x), \lambda \alpha(y) \}, \inf \{ \lambda \alpha(x) + y \} \} \).

Hence \( \lambda \alpha(x) + y \geq \{ \alpha \beta \} \). We prove that \( \lambda \alpha \) is a subspace of \( W \) whenever it is non empty.

\[ V_4(\lambda \alpha(x) + \lambda \alpha(y)) = \inf \{ V_4(\lambda \alpha(x)), V_4(\lambda \alpha(y)) \} \geq \inf \{ \inf \{ V_4(\lambda \alpha(x)), V_4(\lambda \alpha(y)) \}, \inf \{ \lambda \alpha(x) + y \} \} \]

\[ = \inf \{ \{ \lambda \alpha(x), \lambda \alpha(y) \}, \inf \{ \lambda \alpha(x) + y \} \} \]

Hence \( V_4(\lambda \alpha(x) + \lambda \alpha(y)) \geq \inf \{ V_4(\lambda \alpha(x)), V_4(\lambda \alpha(y)) \} \). \( V_4(\lambda \alpha(x)) = \inf \{ \lambda \alpha(x), \lambda \alpha(y) \} \geq \inf \{ \lambda \alpha(x), \lambda \alpha(y) \} \). Thus \( V_4(\lambda \alpha(x)) \geq \lambda \alpha \). Therefore \( A \cap B \) is an \( I \)-vague subspace of \( W \).

Lemma 13: Let \( I \) be complete. If \( \{ A_i : i \in \Delta \} \) is a non empty family of \( I \)-vague subspaces of \( W \), then \( \bigcap_{i \in \Delta} A_i \) is an \( I \)-vague subspace of \( W \).

Proof: Let \( \{ A_i : i \in \Delta \} \) be a non empty family of \( I \)-vague subspaces of \( W \). Let \( A, B \in \{ A_i : i \in \Delta \} \). We prove that \( A \cap B \) is an \( I \)-vague subspace of \( W \).

Let \( x, y \in W \). Then

\[ V_4(x + y) = \inf \{ V_4(x), V_4(y) \} \geq \inf \{ \inf \{ V_4(x), V_4(y) \}, \inf \{ \inf \{ V_4(x), V_4(y) \}, \inf \{ x + y \} \} \} \]

\[ = \inf \{ \{ V_4(x), V_4(y) \}, \inf \{ \inf \{ V_4(x), V_4(y) \}, \inf \{ x + y \} \} \} \]

\[ = \inf \{ \{ V_4(x), V_4(y) \}, \inf \{ \inf \{ V_4(x), V_4(y) \}, \inf \{ x + y \} \} \} \]

Thus \( V_4(x + y) \geq \inf \{ V_4(x), V_4(y) \} \). \( V_4(x) = \inf \{ V_4(x) \} \). \( V_4(x) = \inf \{ V_4(x) \} \). Hence \( V_4(x) \geq \inf \{ V_4(x) \} \). Hence \( V_4(x) \geq V_4(y) \). Therefore \( A \cap B \) is an \( I \)-vague subspace of \( W \).

Lemma 14: Let \( U \neq \emptyset \). The \( I \)-vague characteristic function set of \( U, \chi_U \) is an \( I \)-vague subspace of \( W \) iff \( U \) is a subspace of \( W \).

Proof: Suppose that \( \chi_U \) is an \( I \)-vague subspace of \( W \). Let \( x, y \in U \). Then \( V_4(x) = \{ 1, 1 \} \) and \( V_4(y) = \{ 1, 1 \} \).

Hence \( V_4(x + y) = \{ 1, 1 \} \). So \( x + y \in U \). \( V_4(x + y) = \{ 1, 1 \} \). Therefore \( U \) is a subspace of \( W \).
of $W$. Conversely, suppose that $U$ is a subspace of $W$. Then Consider

$$V_{k_1}(x) = \begin{cases} [1, 1] & \text{if } x \in U \\ [0, 0] & \text{otherwise.} \end{cases}$$

By Lemma 11, $\chi_U$ is an I-vague subspace of $W$.

**Theorem 6:** Let $A$ be an I-vague subspace of a vector space $W$. If $V_A(x-y) = V_A(0)$ for all $x, y \in W$, then $V_A(x) = V_A(y)$.

**Proof:** Let $A$ be an I-vague subspace of a vector space $W$. Suppose that $V_A(x-y) = V_A(0)$ for $x, y \in W$. We prove that $V_A(x) = V_A(y)$. $V_A(x-y) = V_A(0)$ implies that $V_A(y-x) = V_A(0)$.

$$V_A(x) = V_A((x-y) + y).$$

$$\geq \inf\{V_A(x-y), V_A(y)\} = \inf\{V_A(0), V_A(y)\} = V_A(y)$$

Thus $V_A(x) \geq V_A(y)$. Similarly, $V_A(y) = V_A((y-x) + x) \geq \inf\{V_A(y-x), V_A(x)\} = \inf\{V_A(0), V_A(x)\} = V_A(x)$. Thus $V_A(y) \geq V_A(x)$. Hence $V_A(x) = V_A(y)$.

The following example shows that the converse of the above theorem is not true.

**Example 3:** Let $I$ be the unit interval $[0, 1]$ of real numbers. Define $a \oplus b = \min\{1, a + b\}$. With the usual ordering $(I, \oplus, \leq, -)$, it is an involutary DRL-semigroup. Let $W = \mathbb{R}^2$ over $\mathbb{R}$. Then $U = \{(x, y) : x + 2y = 0\}$ is a subspace of $W$. Define the I-vague subspace $A$ of $W$ by

$$V_A(u) = \begin{cases} [1, 1] & \text{if } u \in U \\ [0, 1] & \text{otherwise.} \end{cases}$$

Let $u = (-2, 2)$ and $v = (1, 2)$. $V_A(u) = V_A(v) = [0, 1]$ and $V_A(u-v) = V_A((-3, 0)) = [0, 1] \neq V_A(0)$. Thus $V_A(u) = V_A(v)$ but $V_A(u-v) \neq V_A(0)$.

**Theorem 7:** Let $A$ be an I-vague subspace of a vector space $W$ and $x \in W$. Then $V_A(x+y) = V_A(y)$ for all $y \in W$ iff $V_A(x) = V_A(0)$.

**Proof:** Let $A$ be an I-vague subspace of a vector space $W$ and $x \in W$. Suppose that $V_A(x+y) = V_A(y)$ for all $y \in W$. Take $y = 0$. Hence $V_A(x) = V_A(0)$. Conversely, suppose that $V_A(x) = V_A(0)$. Let $y \in W$. Then $V_A(x+y) \geq \inf\{V_A(x), V_A(y)\} = V_A(y)$. It follows that $V_A(x+y) \geq V_A(y)$.

$$V_A(y) = V_A(-x + x + y) \geq \inf\{V_A(-x), V_A(x+y)\} = \inf\{V_A(x), V_A(0), V_A(x+y)\} = \inf\{V_A(0), V_A(x+y)\} = V_A(x+y)$$

Thus $V_A(y) \geq V_A(x+y)$. It follows that $V_A(x+y) = V_A(y)$.

**Theorem 8:** Let $A$ be an I-vague subspace of a vector space $W$. If $V_A(x-y) = V_A(0)$ for all $x, y \in W$, then $V_A(x) = V_A(y)$.

**Proof:** Let $A$ be an I-vague subspace of a vector space $W$. $V_A(x) = V_A((x-y) + y) \geq \inf\{V_A(x-y), V_A(y)\} = \inf\{V_A(0), V_A(y)\} = V_A(y)$. Similarly, $V_A(y) = V_A((y-x) + x) \geq \inf\{V_A(y-x), V_A(x)\} = \inf\{V_A(0), V_A(x)\} = V_A(x)$. Hence $V_A(x) = V_A(y)$.

**Theorem 9:** Let $W_1$ and $W_2$ be vector spaces over a field $F$, and let $T$ be a linear transformation from $W_1$ into $W_2$. If $A$ is an I-vague subspace of $W_2$, then $T^{-1}(A)$ is an I-vague subspace of $W_1$.

**Proof:** Let $T$ be a linear transformation from $W_1$ into $W_2$ and $A$ be an I-vague subspace of $W_2$.

$$V_{T^{-1}(A)}(\lambda x + \mu y) = V_A(T(\lambda x + \mu y))$$

$$= V_A(\lambda T(x) + \mu T(y))$$

$$\geq \inf\{V_A(\lambda T(x)), V_A(\mu T(y))\}$$

$$\geq \inf\{V_A(T(x)), V_A(T(y))\}$$

$$= V_{T^{-1}(A)}(x, y)$$

This completes the proof.

**Theorem 10:** Let $I$ be complete and infinitely meet distributive. Let $U$ and $V$ be vector spaces over a field $F$ and $T : U \to V$ be a linear transformation. If $A$ is an I-vague subspace of $U$, then $T(A)$ is an I-vague subspace of $V$.

**Proof:** Let $T : U \to V$ be a linear transformation and $A$ be an I-vague subspace of $U$.

$$V_{T(A)}(y_1 + y_2) = \sup\{V_{T(A)}(z) : z \in T^{-1}(y_1 + y_2)\}$$

$$\geq \sup\{V_{T(A)}(z) : z = x_1 + x_2 \text{ where } x_1 \in T^{-1}(y_1) \text{ and } x_2 \in T^{-1}(y_2)\}$$

$$= \sup\{V_A(x_1 + x_2) : x_1 \in T^{-1}(y_1) \text{ and } x_2 \in T^{-1}(y_2)\}$$

$$\geq \inf\{\inf\{V_A(x_1), V_A(x_2)\} : x_1 \in T^{-1}(y_1) \text{ and } x_2 \in T^{-1}(y_2)\}$$

$$= \inf\{\inf\{V_A(x_1), V_A(x_2)\} : x_1 \in T^{-1}(y_1) \text{ and } x_2 \in T^{-1}(y_2)\}$$

since $I$ is infinitely meet distributive

$$= \inf\{V_{T(A)}(y_1), V_{T(A)}(y_2)\}$$

$$V_{T(A)}(y) = \sup\{V_{T(A)}(z) : z \in T^{-1}(y)\}$$

$$= \sup\{V_{T(A)}(z) : T(z) = y\}$$

$$\leq \sup\{V_A(\lambda z) : T(\lambda z) = y \text{ for any } \lambda \in F\}$$

$$= \sup\{V_A(\lambda z) : T(\lambda z) = \lambda y\}$$

$$= \sup\{V_A(u) : T(u) = \lambda y\}$$

$$= V_{T(A)}(\lambda y)$$

This proves the theorem.

**REFERENCES**


