Determine the Solution of Delay Differential Equations using Runge-Kutta Methods with Cubic-Spline Interpolation

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***Abstract*—This paper describes some iterations for term delay in Delay Differential Equation (DDE), which is causing a huge number of iteration calculations. Time-delay was approximated using Cubic-Spline Interpolation, so DDE can rewrite as Differential Equations. Then, Runge-Kutta methods have been used to determine the solution of Differential equations from DDE.**

***Index Terms*—Delay Differential Equations, Cubic-Spline Interpolation, Runge-Kutta Methods, Lipschitz.**

I. Introduction

Delay Differential Equations (DDE) have many uses in the fields of biology, health, chemistry, physics, engineering, and economics. For instance, modeling the human respiratory system in biology, determining nonlinear oscillations in mining, and producing a semi-dynamic system. Therefore, determining the solution of the delay differential equation is the main thing. Several studies using DDE have been researched by Batzel & Tran [1]conducted a study of the stability characteristics of a feedback control system of five differential equations with a time delay for modeling the human respiratory system. Furthermore, Balachandran B [2] examines nonlinear oscillations in mining whose mathematical model is non-linear, non-homogeneous, and is a delay-differential system with a time-periodic coefficient.

Some research on DDE has led to finding a solution to this equation. Among them, Karakog & Bereketoglu [3] use the differential transformation method to determine the DDE solution. Next Alomari, Noorani, & Nazar [4] use the average homotopy method. Several other studies discuss DDE solutions with various numerical methods such as the spectral method [5], the multistep block method [6], and the Predictor-Corrector method [7]. Overall these methods have weaknesses related to the use of time-delay in the DDE equation. The use of term delay causes many calculations per iteration and less time efficiency at the time of calculation. So Ismail, Al-Khasawneh, Lwin, & Suleiman [8] use Hermite interpolation as an estimated DDE.

This research aims to find a new iteration method using the Runge-Kutta method to determine the solution of DDE by estimating the term-delay in the equation using Cubic-Splines interpolation.

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Manuscript received December 10, 2019; accepted October 3, 2022.

II. Problem Description

Consider the form of Delay Differential Equations (DDE) with initial value problems

$$\begin{cases} y'(t) = f(t, y, y(t-\alpha)), \quad t \geq 0, \quad y(t) = y_0, \\ y(t) = \varphi(t), \quad t_0 - \alpha \leq t \leq t_0 \end{cases} \tag{1}$$

with $y(t)$ n-vector valued function, $\alpha > 0$ is a constant delay, and $\varphi(t)$ is an initial function, which is assumed piecewise continuous in the interval $t_0 - \alpha \leq t \leq t_0$, for the case $\varphi(t) \neq t_0$ is included.

A solution for (1) is a piecewise differentiable function $y(t)$ defined on the interval $t_0 - \alpha \leq t \leq t_f(t_f > t_0)$ which continues for all $t > t_0$ and satisfies differential equations. The DDE only satisfies left and right limit points $t_0 + \alpha$ and point $\xi + \alpha$, where ξ is some jump point in φ. But, the smoothness of f and φ causes its solutions do not smooth, however, $\varphi(t_0) = y_0$ holds.

Lemma 1: [9] Let the function $f(t, y, z)$ and $\varphi(t)$ be analytic with respect to all arguments, and let $y(t)$ denote a solution of defined on an interval D_y. Then the following properties hold :

1) If $\varphi(t_0) \neq y_0$ and $t_i = t_0 + i\alpha \in D_y (i \in N)$, then $y(t)$ has $i-1$ continuous derivatives at t_i, and, in general, $y^i(x)$ has a jump discontinuity at t_i.
2) If $\varphi(t_0) \neq y_0$ and $t_i = t_0 + i\alpha \in D_y (i \in N)$, then $y(t)$ has i continuous derivatives at t_i, and, in general, $y^{i+1}(x)$ has a jump discontinuity at t_i.

Lemma 2: [9]

1) If ξ is a jump discontinuity of $\varphi(t)$, then $y(t)$ is of class C^{i-1} at $i = \xi + i * \alpha$, but, in general, $y(t)$ is not of the class C^i at $i(i = 1, 2, \dots)$.
2) If ξ is a jump discontinuity of $f(t, y, z)$, i.e. for sufficiently small $\epsilon > 0$:

$$f(t, y, z) = \begin{cases} f_1(t, y, z), \xi - \epsilon < t < \xi \\ f_2(t, y, z), \xi < t < \xi + \epsilon \end{cases}$$

holds, where f_1 and f_2 are analytic on $[\xi - \epsilon, \xi] \times R^{2n}$ and ξ, $\xi \in \mathbb{R}^{2n}$, respectively, then $y(t)$ is of the class C^i at $i = \xi + i\alpha$, but, in general, $y(t)$ is not of the class C^{i+1} at $\xi_i (i = 0, 1, 2, \dots)$. The jump discontinuities $t_i = t_0 + i\alpha$ caused by the initial point t_0 are called primary discontinuities. The jump discontinuities $\xi_i = \xi + i\alpha$ caused by discontinuities of φ or f are called secondary discontinuities.

III. Numerical Method

One way to solve the DDE initial value problem is to develop a one-step method, which is by combining the one-step method for Ordinary Differential Equations (ODE) and approximating the delayed term formula. Let

$$\begin{cases} y'(t) & = f(t, y(t), y(t-\alpha(t, y(t)))), t_0 \le t \le t_f \\ y(t) & = \varphi(t), t \le t_0 \end{cases} \tag{2}$$

with $f : [t_0, t_f] \times R^n \times R^n \to R^n$ which is based on the continuous ODE method and has steps that are jump over the mesh $\triangle = \{t_0, t_1, ..., t_n, ..., t_N = t_f\}$. After approximation y_n was obtained at time t_n, next, step-$(n-1)$ was completed by using equation (3.2.3) in [10], the equation

$$\begin{cases} w'_{n-1}(t) & = f(t, w_{n+1}(t), x(t-\alpha(t, w_{n+1}(t)))) \\ w'_{n+1}(t_n) & = y_n \end{cases} \tag{3}$$

with

$$x(s) = \begin{cases} \varphi(s), & s \le t_0 \\ \eta(s), & t_0 \le s \le t_n \\ w_{n+1}(s), & t_n \le s \le t_{n+1} \end{cases} \tag{4}$$

and $\eta(s)$ is Cubic Spline Interpolation. To solve the interpolation of the equation (4), the steps used must statisfies the conditions of consistency from f, namely Lipschitz's condition [11]:

$$|f(t,y,z) - f(t,y',z')| \le L_1 |y-y'| + L_2 |z-z'|,$$

for all $t \in [t_0, t_f], y, y', z, z' \in R$. Let $I = [t_0, t_f]$ was partition into N partition with equal range and denoted by

$$I_i = [t_{i-1}, t_i], i = 1, 2, ..., N$$

and $\alpha = mh$, $t_i - \alpha \in I_{i-m} = [t_{i-1-m}, t_{i-m}]$. Then, for every subinterval I_i, cubic spline function $\eta(t)$ can be written in the form

$$\eta(t) = \begin{cases} \bar{b}^2(2b+1)\eta_{i-1}^{(0)} + \bar{b}^2 b\eta_{i-1}^{(1)} + \\ \qquad b^2(2\bar{b}+1)\eta_i^{(0)} - b^2\bar{b}\eta_i^{(1)}, & \forall t \in I_i \\ \rho(t), & \alpha \le t \le 0 \end{cases} \tag{5}$$

with $b = \frac{t-t_{i-1}}{h} \in [0,1], \bar{b} = 1-b, \eta_i^0 = \eta(ti), \eta_i^{(1)} = hS'^{(ti)}, i = 0, \ldots, N$. Subtitution $t_{i-1+\alpha}$ and $t_{i-1+\beta}$ into (5), respectively, obtained

$$\begin{aligned} \begin{bmatrix} \eta(t_{i-1}+h\gamma) \\ \eta(t_{i-1}+h\delta) \end{bmatrix} &= \begin{bmatrix} \bar{\gamma}^2(2\gamma+1) & \bar{\gamma}^2\gamma \\ \bar{\delta}^2(2\delta+1) & \bar{\delta}^2\delta \end{bmatrix} \begin{bmatrix} \eta_{i-1}^{(0)} \\ \eta_{i-1}^{(1)} \end{bmatrix} \\ &+ \begin{bmatrix} \bar{\gamma}^2(2\gamma+1) & -\bar{\gamma}^2\gamma \\ \bar{\delta}^2(2\delta+1) & -\bar{\delta}^2\delta \end{bmatrix} \begin{bmatrix} \eta_i^{(0)} \\ \eta_i^{(1)} \end{bmatrix} \\ &= C(\gamma,\delta) \begin{bmatrix} \eta_{i-1}^{(0)} \\ \eta_{i-1}^{(1)} \end{bmatrix} + D(\gamma,\delta) \begin{bmatrix} \eta_i^{(0)} \\ \eta_i^{(1)} \end{bmatrix}, \ \forall \gamma, \delta \in [0,1], \end{aligned}$$

with $\overline{\gamma} = 1-\gamma$, $\overline{\delta} = 1-\delta$. Next, find derivative from equation (5) in both sides, such that obtained $h\eta'(t) = \bar{b}(-6b)\eta_{i-1}^{(0)} + \bar{b}(1-3b)\eta_{i-1}^{(1)} + b(6\bar{b}\eta_i^{(0)} + b(1-3\bar{b})\eta_i^{(1)}$. Let

$$M_i = \eta''(t_i), \qquad i = 1, 2, \ldots, n$$

because $\eta(t)$ in every interval $[t_i, t_{i+1}] \longrightarrow \eta"(t)$ is linear, such that

$$\eta"(t) = \frac{(t_{i+1-t})M_i + (t-t_i)M_{i+1}}{h_i}, \quad i = 0, 1, \cdots, n$$

with $h_i = t_{i+1} - t_i$. Then, $\eta"(t)$ was integrated twice, such that

$$\begin{aligned} \eta(t) = & \frac{(t_{i+1-t})^3 M_i + (t-t_i)^3 M_{i+1}}{6h_i} + \\ & \frac{(t_{i+1-t})f_i + (t-t_i)f_{i+1}}{h_i} - \\ & \frac{h_i}{6}(t_{i+1-t})M_i + (t-t_i)M_{i+1}. \end{aligned} \tag{6}$$

Equation (6) causes $\eta(t)$ continue in $[t_0, t_f]$ and statisfy interpolation condition $\eta(t) = f$.

In this paper, a one-step method to solve the solution of DDE equation was described. The basic one-step method uses the Runge-Kutta method of order 5 and approximates the term delay by using cubic spline interpolation, obtained

$$Y_{n+1}^i = y_n + h_{n+1}\sum_{j=1}^{s} a_{ij}k_i, i = 1, \cdots, s,$$

$$k_i = f\left(t_{n+1}^j, Y_{n+1}^j, \eta\left(t_{n+1}^j - \alpha\left(t_{n+1}^j, Y_{n+1}^j\right)\right)\right).$$

IV. Conclusions

To get the solution from DDE, the following algorithms from this method are:

1) Determine the initial value of the function.
2) Partition intervals $[t0, tf]$ into sub-intervals.
3) Determine $y(t)$ using the first subinterval.
4) Determine k_i using the Runge-Kutta method for DDE. Then estimate the term-delay using cubic spline interpolation.

Acknowledgment

The author would like to thank you for the support of the Ministry of Research, Technology and Higher Education (KEMENRISTEK DIKTI) of the Republic of Indonesia, LLDIKTI Region X and the Institute of Research and Community Service (LPPM) of the Universitas Islam Riau.

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