The Classification of Diffeomorphism Classes of Real Bott Manifolds

Admi Nazra

***Abstract*—A real Bott manifold (RBM) is obtained as the orbit space of the $n-torus$ T^n by a free action of an elementary abelian $2-group$ $(\mathbb{Z}_2)^n$. This paper deals with the classification of some particular types of RBMs of dimension n, so that we know the number of diffeomorphism classes in such RBMs.**

***Index Terms*—Real Bott manifolds, orbit space, diffeomorphism classes, Seifert fiber space.**

I. Introduction

KAMISHIMA et al. [1], [2] defined a real Bott manifold of dimension n (RBM_n) as the total space B_n of the sequence of $\mathbb{R}P^1$-bundles

$$B_n \to B_{n-1} \to \cdots \to B_2 \to B_1 \to \{\text{a point}\} \qquad (1)$$

starting with a point, where each $\mathbb{R}P^1$-bundle $B_i \to B_{i-1}$ is the projectivization of the Whitney sum of a real line bundle L_i and the trivial line bundle over B_{i-1}. Then, from the viewpoint of group actions, it was explained that a RBM_n is the quotient of the torus of dimension n, $T^n = S^1 \times \cdots \times S^1$ by the product $(\mathbb{Z}_2)^n$ of cyclic group of order 2. Such RBM_n can be expressed by an upper triangular matrix A of size n (called a Bott matrix of size n, BM_n) whose entries are either 1 or 0 except the diagonal entries which are 0. Each row of the BM_n A express the free action of $(\mathbb{Z}_2)^n$ on T^n and the orbit space $M_n(A) = T^n/(\mathbb{Z}_2)^n$ is the RBM_n. In fact, $M_n(A)$ is a Riemannian flat manifold (compact Euclidean space form). To classify RBM_ns, we can apply the Bieberbach Theorem [3] and by this theorem, it was obtained in [1], [4] the classification of RBMs up to dimension 4.

Kamishima and Nazra proved in [2] that every RBM_n $M_n(A)$ admits an injective Seifert fibred structure which has the form $M_n(A) = T^k \times_{(\mathbb{Z}_2)^s} M(B)$, that is there is a k-torus action on $M_n(A)$ whose quotient space is an $(n-k)$-dimensional real Bott orbifold $M_{n-k}(B)/(\mathbb{Z}_2)^s$ by some $(\mathbb{Z}_2)^s$-action $(1 \leq s \leq k)$. Moreover, they have proved the smooth rigidity that two RBM_ns $M_n(A_1)$ and $M_n(A_2)$ are diffeomorphic if and only if the corresponding actions $((\mathbb{Z}_2)^{s_1}, M_{n-k_1}(B_1))$ and $((\mathbb{Z}_2)^{s_2}, M_{n-k_2}(B_2))$ are equivariantly diffeomorphic. By the above rigidity we can determine the diffeomorphism classes of higher dimensional $RBMs$ when the low dimensional ones with $(\mathbb{Z}_2)^s$-actions are classified. RBMs up to dimension 5 have been classified (see [5], [6]).

This paper aims to study the number of diffeomorphism classes in some particular types of RBM_ns.

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II. Preliminaries

In this section, we shall review some concepts from [2] related to the RBM.

A. Seifert fiber space

In a BM_n A, each i-th row defines a $\mathbb{Z}_2$-action on T^n by

$$g_i(z_1, z_2, \ldots, z_n) = (z_1, \ldots, z_{i-1}, -z_i, \tilde{z}_{i+1}, \ldots, \tilde{z}_n)$$

where $\tilde{z}_m$ is either z_m or $\bar{z}_m$ depending on whether (i, m)-entry $(i < m)$ is 0 or 1 respectively while (i, i)-(diagonal) entry 0 acts as $z_i \to -z_i$. Note that $\bar{z}$ is the conjugate of the complex number $z \in S^1$. It is always trivial; $z_m \to z_m$ whenever $m < i$. Here $(z_1, \ldots, z_n)$ are the standard coordinates of the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ whose universal covering is the n-dimensional Euclidean space $\mathbb{R}^n$. The projection $p\colon \mathbb{R}^n \to T^n$ is denoted by

$$p(x_1, \ldots, x_n) = (e^{2\pi \mathbf{i} x_1}, \ldots, e^{2\pi \mathbf{i} x_n}) = (z_1, \ldots, z_n).$$

Those $g_1, \ldots, g_n$ constitute the generators of $(\mathbb{Z}_2)^n$. In fact, $(\mathbb{Z}_2)^n$ acts freely on T^n such that the orbit space $M_n(A) = T^n/(\mathbb{Z}_2)^n$ is a smooth compact n-dimensional manifold. In this way, given a BM_n A, we obtain a free action of $(\mathbb{Z}_2)^n$ on T^n.

Let $\pi(A) = \langle \tilde{g}_1, \ldots, \tilde{g}_n \rangle$ be the lift of $(\mathbb{Z}_2)^n = \langle g_1, \ldots, g_n \rangle$ to $\mathbb{R}^n$. Then, we get

$$\tilde{g}_i(x_1, x_2, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \frac{1}{2} + x_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n)$$

where $\tilde{x}_m$ is either x_m or $-x_m$. One can see that $\pi(A)$ acts properly discontinuously and freely on $\mathbb{R}^n$ as Euclidean motions. Note that $\pi(A)$ is a Bieberbach group which is a discrete uniform subgroup of the Euclidean group $\mathbb{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$ (cf. [3]). It follows that

$$\mathbb{R}^n/\pi(A) = T^n/(\mathbb{Z}_2)^n = M_n(A).$$

Now, we consider the following moves (**I, II, III**) to A under which the diffeomorphism class of RBM_n $M_n(A)$ does not change.

I If the j-th column has all 0-entries for some $j > 1$, then interchange the j-th column and the $(j-1)$-th column. Next, interchange the j-th row and the $(j-1)$-th row.

We perform move **I** iteratively to get a BM_n A'.

$$A = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}, A' = \begin{pmatrix} \mathrm{O}_k & C \\ 0 & B \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}.$$

O_k is a $k \times k$ zero matrix $(1 \leq k \leq n)$ and we call it a block zero matrix of size k.

Note the following.

(1) O_k is a maximal block of zero matrix.

(2) As B is an $(n-k)$-dimensional Bott matrix, we obtain a real Bott manifold $M_{n-k}(B) = T^{n-k}/(\mathbb{Z}_2)^{n-k}$.

(3)
$$M_n(A) = \frac{T^k \times T^{n-k}}{(\mathbb{Z}_2)^k \times (\mathbb{Z}_2)^{n-k}} = T^k \underset{(\mathbb{Z}_2)^k}{\times} M_{n-k}(B) = M_n(A').$$

(4) The matrix C corresponds to $(\mathbb{Z}_2)^k$ -action on T^{n-k}.

II For an m-th row $(1 \leq m \leq k)$ whose entries in C are all zero, divide $T^k \times M_{n-k}(B)$ by the corresponding $\mathbb{Z}_2$-action.

III If there are two rows, p-th row and ℓ-th row $(1 \leq p < \ell \leq k)$, having the common entries in the C, then compose the $\mathbb{Z}_2$-action of p-th row and ℓ-th row and divide $T^k \times M_{n-k}(B)$ by $\mathbb{Z}_2$-action.

By using **II, III**, the quotient is again diffeomorphic to $T^k \times_{(\mathbb{Z}_2)^k} M_{n-k}(B)$ but consequently the $(\mathbb{Z}_2)^k$-action is reduced to the effective $(\mathbb{Z}_2)^s$-action on $T^k \times M_{n-k}(B)$. Therefore A' reduces to

$$A'' = \begin{pmatrix} 0_{k-s} & 0 & 0 \\ 0 & 0_s & * \\ 0 & 0 & B \end{pmatrix} \tag{2}$$

in which $M_n(A') = T^k \times_{(\mathbb{Z}_2)^k} M_{n-k}(B) = \frac{T^{k-s} \times T^s \times M_{n-k}(B)}{(\mathbb{Z}_2)^{k-s} \times (\mathbb{Z}_2)^s} = M_n(A'')$. Since $(\mathbb{Z}_2)^{k-s}$ acts trivially on $T^s \times M_{n-k}(B)$, we have $M_n(A'') \cong T^k \times_{(\mathbb{Z}_2)^s} M_{n-k}(B)$.

Hereinafter, we write $M_n(A)$ in place of $M_n(A'')$.

Remark 1: Concerning $*$ in (2), the group $(\mathbb{Z}_2)^s = \langle g_{k-s+1}, \ldots, g_k \rangle$ acts on $T^k \times M_{n-k}(B)$ by

$$\begin{aligned} &g_i(z_1, \ldots, z_{k-s+1}, \ldots, z_k, [z_{k+1}, \ldots, z_n]) \\ &= (z_1, \ldots, z_{k-s+1}, \ldots, -z_i, \ldots, z_k, [\tilde{z}_{k+1}, \ldots, \tilde{z}_n]) \end{aligned} \tag{3}$$

where $\tilde{z} = \bar{z}$ or z. So there induces an action of $(\mathbb{Z}_2)^s$ on $M_{n-k}(B)$ by

$$g_i([z_{k+1}, \ldots, z_n]) = [\tilde{z}_{k+1}, \ldots, \tilde{z}_n]. \tag{4}$$

Moreover in [2], it was obtained the following theorem.

Theorem 1 (Structure): For a RBM_n $M_n(A)$, there is a maximal T^k-action $(k \geq 1)$ such that $M_n(A) = T^k \times_{(\mathbb{Z}_2)^s} M_{n-k}(B)$ is an injective Seifert fiber space over the $(n-k)$-dimensional real Bott orbifold $M_{n-k}(B)/(\mathbb{Z}_2)^s$;

$$T^k \to M_n(A) \to M_{n-k}(B)/(\mathbb{Z}_2)^s. \tag{5}$$

There exist a central extension of the fundamental group $\pi(A)$ of $M_n(A)$:

$$1 \to \mathbb{Z}^k \to \pi(A) \to Q_B \to 1 \tag{6}$$

such that

(i) $\mathbb{Z}^k$ is the maximal central free abelian subgroup

(ii) The induced group Q_B is the semidirect product $\pi(B) \rtimes (\mathbb{Z}_2)^s$ for which $\mathbb{R}^{n-k}/\pi(B) = M_{n-k}(B)$.

See [2] for the proof.

Using this theorem, a RBM_n $M_n(A)$ which admits a maximal T^k-action $(k \geq 1)$ can be created from an RBM_{n-k} $M_{n-k}(B)$ by a $(\mathbb{Z}_2)^s$-action, and the corresponding BM_n A has the form as in (2) above.

B. Affine maps between real Bott manifolds

Next, to check whether two RBMs are diffeomorphic, we can apply the following theorem.

Theorem 2 (Rigidity): Suppose that $M_n(A_1)$ and $M_n(A_2)$ are RBM_ns and $1 \to \mathbb{Z}^{k_i} \to \pi(A_i) \to Q_{B_i} \to 1$ is the associated group extensions $(i = 1, 2)$. Then, the following are equivalent:

(i) $\pi(A_1)$ is isomorphic to $\pi(A_2)$.

(ii) There exists an isomorphism of $Q_{B_1} = \pi(B_1) \rtimes (\mathbb{Z}_2)^{s_1}$ onto $Q_{B_2} = \pi(B_2) \rtimes (\mathbb{Z}_2)^{s_2}$ preserving $\pi(B_1)$ and $\pi(B_2)$.

(iii) The action $((\mathbb{Z}_2)^{s_1}, M_{n-k}(B_1))$ is equivariantly diffeomorphic to the action $((\mathbb{Z}_2)^{s_2}, M_{n-k}(B_2))$.

See [2] for the proof. Here Bott matrices A_1 and A_2 are created from B_1 and B_2 respectively.

Note that two RBM_ns $M_n(A_1)$ and $M_n(A_2)$ are diffeomorphic if and only if $\pi(A_1)$ is isomorphic to $\pi(A_2)$ by the Bieberbach theorem [3]. Moreover, by Theorem 1 and 2 we have,

Remark 2: Let RBM_ns $M_n(A_i) = T^{k_i} \times_{(\mathbb{Z}_2)^{s_i}} M_{n-k_i}(B_i)$ $(i = 1, 2)$. If $M_n(A_1)$ and $M_n(A_2)$ are diffeomorphic then the following hold.

(i) $k_1 = k_2$.

(ii) $M_{n-k_1}(B_1)$ and $M_{n-k_2}(B_2)$ are diffeomorphic.

(iii) $s_1 = s_2$.

If two RBMs have the same maximal T^k-action, then the quotients $((\mathbb{Z}_2)^{s_i}, M_{n-k_i}(B_i))$ are compared. So, what we have to do next is to distinguish the $(\mathbb{Z}_2)^{s_i}$-action on $M_{n-k_i}(B_i)$ when it is the case that $s_1 = s_2 = s$ and $M_{n-k_1}(B_1)$ is diffeomorphic to $M_{n-k_2}(B_2)$.

C. Type of fixed point set

Note that from (4), the action of $(\mathbb{Z}_2)^s$ on $M_{n-k}(B)$ is defined by $\alpha[(z_1, \ldots, z_{n-k})] = [\alpha(z_1, \ldots, z_{n-k})] = [(\tilde{z}_1, \ldots, \tilde{z}_{n-k})]$ for $\alpha \in (\mathbb{Z}_2)^s$ and $\tilde{z} = z$ or $\bar{z}$. Since $M_{n-k}(B) = T^{n-k}/(\mathbb{Z}_2)^{n-k}$, the action $\langle \alpha \rangle$ lifts to a linear (affine) action on T^{n-k} naturally: $\alpha(z_1, \ldots, z_{n-k}) = (\tilde{z}_1, \ldots, \tilde{z}_{n-k})$. Then, the fixed point set is characterized by the equation: $(\tilde{z}_1, \ldots, \tilde{z}_{n-k}) = g(z_1, \ldots, z_{n-k})$ for some $g \in (\mathbb{Z}_2)^{n-k}$. It is also an affine subspace of T^{n-k}. So the fixed point sets of $(\mathbb{Z}_2)^s$ are affine subspaces in $M_{n-k}(B)$.

Let B be the Bott matrix as in above. By a repetition of move **I**, B has the form

$$B = \begin{pmatrix} 0_{b_2} & C_{23} & \ldots & \ldots & C_{2\ell} \\ & 0_{b_3} & C_{34} & \ldots & C_{3\ell} \\ & & \ddots & & \ldots \\ 0 & & & 0_{b_{\ell-1}} & C_{(\ell-1)\ell} \\ & & & & 0_{b_\ell} \end{pmatrix} \tag{7}$$

where rank $B = b_2 + \cdots + b_\ell = n - k$ ($b_i \geq 1$), C_{jt} ($j = 2, \ldots, \ell - 1$, $t = 3, \ldots, \ell$) is a $b_j \times b_t$ matrix.

Note that by the Bieberbach theorem (cf. [3]), if f is an isomorphism of $\pi(A_1)$ onto $\pi(A_2)$, then there exists an affine element $g = (h, H) \in \mathrm{A}(n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$ such that

$$f(r) = grg^{-1} \ (\forall r \in \pi(A_1)). \tag{8}$$

Recall that if $M_n(A_1)$ is diffeomorphic to $M_n(A_2)$ then $M_{n-k}(B_1)$ is diffeomorphic to $M_{n-k}(B_2)$. This implies that B_1 and B_2 have the form as in (7).

Using (8) and according to the form of B in (7) we obtain that

$$g = \left(\begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \vdots \\ \mathbf{h}_\ell \end{pmatrix}, \begin{pmatrix} H_1 & & & \\ & H_2 & & 0 \\ 0 & & \ddots & \\ & & & H_\ell \end{pmatrix} \right) \tag{9}$$

where $\mathbf{h}_i$ is an $b_i \times 1$ (s_i=rank I_i) column matrix ($\mathbf{h}_1$ is a $k \times 1$ column matrix), $H_i \in \mathrm{GL}(b_i, \mathbb{R})$ ($i = 2, \ldots, \ell$), $H_1 \in \mathrm{GL}(k, \mathbb{R})$ (see Remark 3.2 [2]).

Let $\bar{f} \colon Q_{B_1} \to Q_{B_2}$ be the induced isomorphism from f (cf. Theorem 2). Now the affine equivalence $\bar{g} \colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ has the form

$$\bar{g} = \left(\begin{pmatrix} \mathbf{h}_2 \\ \vdots \\ \mathbf{h}_\ell \end{pmatrix}, \begin{pmatrix} H_2 & & 0 \\ & \ddots & \\ 0 & & H_\ell \end{pmatrix} \right) \tag{10}$$

which is equivariant with respect to $\bar{f}$. The pair $(\bar{f}, \bar{g})$ induces an equivariant affine diffeomorphism $(\hat{f}, \hat{g}) \colon ((\mathbb{Z}_2)^s, M_{n-k}(B_1)) \to ((\mathbb{Z}_2)^s, M_{n-k}(B_2))$.

Let $\mathrm{rank} H_i = b_i$ ($i = 2, \ldots, \ell$). (Note that $b_2 + \cdots + b_\ell = n - k$.) Since $M_{n-k}(B_1) = T^{n-k}/(\mathbb{Z}_2)^{n-k}$, $\bar{g}$ induces an affine map $\tilde{g}$ of T^{n-k}. Put $X_{b_2} = \begin{pmatrix} x_1 \\ \vdots \\ x_{b_2} \end{pmatrix}, \ldots, X_{b_\ell} = \begin{pmatrix} x_{b_{\ell'}+1} \\ \vdots \\ x_{b_{\ell'}+b_\ell} \end{pmatrix}$, $w_{b_i} = p(X_{b_i}) \in T^{b_i}$ ($i = 2, \ldots, \ell$), $b_{\ell'} = b_2 + \cdots + b_{\ell-1}$. Since $\tilde{g}p = p\bar{g}$, $\tilde{g}({}^t w_{b_2}, \ldots, {}^t w_{b_\ell}) = ({}^t w'_{b_2}, \ldots, {}^t w'_{b_\ell})$ where $w'_{b_i} = p(\mathbf{h}_i + H_i X_{b_i}) \in T^{b_i}$. That is, $\tilde{g}$ preserves each T^{b_i} of $T^{n-k} = T^{b_2} \times \cdots \times T^{b_\ell}$, so does $\hat{g}$ on

$M_{n-k}(B_1) =$
$\{[z_1, \ldots, z_{b_2}; z_{b_2+1}, \ldots, z_{b_2+b_3}; \ldots\ldots; z_{b_{\ell'}+1}, \ldots, z_{b_{\ell'}+b_\ell}]\}$.

We say that $\hat{g}$ *preserves the type* $(b_2, \ldots, b_\ell)$ of $M_{n-k}(B_1)$. As $\hat{g}$ is $\hat{f}$-equivariant, it also preserves the type corresponding to the fixed point sets between $((\mathbb{Z}_2)^s, M_{n-k}(B_1))$ and $((\mathbb{Z}_2)^s, M_{n-k}(B_2))$.

Proposition 1: The $(\mathbb{Z}_2)^s$-action on $M_{n-k}(B)$ is distinguished by the number of components and types of each positive dimensional fixed point subsets.

See [2] for the proof.

Definition 1: We say that two Bott matrices A and A' are *equivalent* (denoted by $A \sim A'$) if $M_n(A)$ and $M_n(A')$ are diffeomorphic.

III. CLASSIFICATION OF PARTICULAR TYPES OF RBM_nS

In this part, we will review some results from [6] and prove some new results regarding the classification of certain n-dimensional real Bott manifolds in order to obtain how many diffeomorphism classes of some particular types of RBM_ns.

Proposition 2: [6] There are 4 diffeomorphism classes of RBM_ns ($n \geq 4$) which admit the maximal T^{n-2}-actions (i.e. $s = 1, 2$):

$$M_n(A) = T^{(n-2)} \underset{(\mathbb{Z}_2)^s}{\times} M_2(B).$$

Proposition 3: [6] The diffeomorphism class is unique for the RBM of the form $M_n(A) = T^k \underset{\mathbb{Z}_2}{\times} T^{n-k}$ for any k ($1 \leq k \leq n-1$). In particular, if $k = n$ then $M_n(A) = T^n$.

Remark 3: By Proposition 3, for $n \geq 2$ there are n distinct diffeomorphism classes of RBM_ns $M_n(A) = T^k \times_{\mathbb{Z}_2} T^{n-k}$ ($1 \leq k \leq n$).

Corollary 1: [6] If the RBM $M(A) = S^1 \times_{\mathbb{Z}_2} M(B)$ where $M(B) = T^k \times_{\mathbb{Z}_2} S^1$, then for any $k \geq 1$ there is only one diffeomorphism class.

Remark 4: By Corollary 1, for $n \geq 3$ there are $n-2$ distinct diffeomorphism classes of RBM_ns $M_n(A) = T^k \times_{\mathbb{Z}_2} M_{n-k}(B)$ ($k = 1, \ldots, n-2$) where $M_{n-k}(B) = T^{k'} \times_{\mathbb{Z}_2} S^1$ ($k' = n-k-1$).

Corollary 2: [6] Let $M(A)$ be a real Bott manifold which fibers S^1 over the real Bott manifold $M(B)$ for which $M(B)$ is $T^k \times_{(\mathbb{Z}_2)^s} \mathrm{K}$ ($k \geq 2$). Here K is Klein bottle. Then the number of diffeomorphism classes of such $M(A)$ is 3.

Remark 5: By Corollary 2, for $n \geq 5$ there are $3(n-4)$ distinct diffeomorphism classes of RBM_ns $M_n(A) = T^k \times_{\mathbb{Z}_2} M_{n-k}(B)$ ($k = 1, \ldots, n-4$) where $M_{n-k}(B) = T^{k'} \times_{(\mathbb{Z}_2)^s} \mathrm{K}$ ($k' = n-k-2 \geq 2,\ s = 1, 2$).

Corollary 3: [6] Let $M(A)$ be a real Bott manifold which fibers S^1 over the real Bott manifold $M(B)$ for which $M(B)$ is $T^k \times_{(\mathbb{Z}_2)^s} T^2$ ($k \geq 2$). Then the number of diffeomorphism classes of such $M(A)$ is 3.

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Proposition 4: [6] Let $M(A)$ be a real Bott manifold which fibers S^1 over the real Bott manifold $M(B)$ where $M(B) = S^1 \times_{\mathbb{Z}_2} T^k$ ($k \geq 2$), then the diffeomorphism classes of such $M(A)$ is $[\frac{k}{2}] + 1$. Here $[x]$ is the integer part of x.

Remark 7: By Proposition 4, for $n \geq 4$ there are $\sum_{k'=2}^{n-2}([\frac{k'}{2}] + 1)$ distinct diffeomorphism classes of RBM_ns $M_n(A) = T^k \times_{\mathbb{Z}_2} M(_{n-k}B)$ ($k = 1, \ldots, n-3$) where $M_{n-k}(B) = S^1 \times_{\mathbb{Z}_2} T^{k'}$ ($k' = n-k-1 \geq 2$).

Proposition 5: For any $k \geq 1$ and $m \geq 2$ ($n - 3 \geq k + m = t \geq 3$), there are $[\frac{n-t}{2}] + 1$ diffeomorphism classes in RBM_ns $M_n(A) = T^k \times_{\mathbb{Z}_2} M_{n-k}(B)$, where $M_{n-k}(B) = T^m \times_{\mathbb{Z}_2} T^{n-k-m}$.

Proof: Similar with the proof of Proposition 4 (see [6]). ∎

Remark 8: By Proposition 5, for $n \geq 6$ there are

$$\sum_{k=1}^{n-5} \sum_{t=k+2}^{n-3} ([\frac{n-t}{2}] + 1)$$

distinct diffeomorphism classes of RBM_ns $M_n(A) = T^k \times_{\mathbb{Z}_2} M_{n-k}(B)$ $(k = 1, \ldots, n-5)$ where $M_{n-k}(B) = T^m \times_{\mathbb{Z}_2} T^{n-t}$ $(m \geq 2,\ n-3 \geq t \geq 3)$.

Proposition 6: [6] Let $M_n(A) = S^1 \times_{\mathbb{Z}_2} M_{n-k}(B)$ be a RBM_n. Suppose that B is either one of the list in (11). Then $M_{n-k}(B)$ are diffeomorphic to each other and the number of diffeomorphism classes of such RBM_ns $M_n(A)$ above is $(k+1)2^{n-k-3}$ $(k \geq 2,\ n-k \geq 3)$.

$$B_1 = \begin{pmatrix} 0 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ & 0 & 1 & \cdots & \cdots & \cdots & 1 \\ & & \ddots & & & & \vdots \\ & & & 0 & 1 & \cdots & 1 \\ \mathbf{0} & & & & & O_k & \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & 0 & 1 & \cdots & \cdots & \cdots & 1 \\ & & \ddots & & & & \vdots \\ & & & 0 & 1 & \cdots & 1 \\ \mathbf{0} & & & & & O_k & \end{pmatrix}, \ldots,$$

$$B_{n-k-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & & \ddots & & & & & \vdots \\ & & & 0 & 1 & 0 & \cdots & 0 \\ & & & & 0 & 1 & \cdots & 1 \\ \mathbf{0} & & & & & & O_k & \end{pmatrix},$$

$$B_{n-k} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & & \ddots & & & & & & & \vdots \\ & & & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ & & & & 0 & 1 & 1 & 0 & \cdots & 0 \\ & & & & & 0 & 1 & 0 & \cdots & 0 \\ & & & & & & 0 & 1 & \cdots & 1 \\ \mathbf{0} & & & & & & & & O_k & \end{pmatrix}, \ldots,$$

$$B_{n-k+(n-k-4)} = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ & 0 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ & & \ddots & & \vdots & \vdots & & \vdots \\ & & & 0 & 1 & 0 & \cdots & 0 \\ & & & & 0 & 1 & \cdots & 1 \\ \mathbf{0} & & & & & & O_k & \end{pmatrix}. \tag{11}$$

Remark 9: By Proposition 6, for $n \geq 5$ there are

$$\sum_{\ell=5}^{n} \sum_{k=2}^{\ell-3} (k+1)2^{\ell-k-3}$$

distinct diffeomorphism classes of RBM_ns $M_n(A) = T^{k'} \times_{\mathbb{Z}_2} M_{n-k'}(B)$ $(k' = 1, \ldots, n-4)$ where B is either one of the list in (11).

Now we consider the other type of real Bott manifolds.

Proposition 7: Let $M_n(A) = T^k \times_{(\mathbb{Z}_2)^2} T^\ell$, $(n = k + \ell \geq 5, \ell \geq 3)$ be a RBM_n. Then the number of diffeomorphism classes of such $M_n(A)$ is

$$\sum_{\ell=3}^{n-2} \left(\left[\frac{\ell}{2}\right] + \sum_{x=1}^{[\frac{\ell}{3}]} \left(\left[\frac{\ell - x}{2}\right] - (x-1) \right) \right).$$

Proof: The proposition follows from Lemmas 1, 2 below. ■

Lemma 1: Let $M_n(A)$ be an RBM_n $(n \geq 5)$ corresponding to the Bott matrix

$$A = \left(\begin{array}{c|cccccc} & & & 0 & & & \\ O_k & 0 & \ldots & 0 & 1 & \ldots & 1 \\ & 1 & \ldots & 1 & 0 & \ldots & 0 \\ \hline 0 & & & O_\ell & & & \end{array} \right) \quad (\ell \geq 3). \tag{14}$$

Then the number of diffeomorphism classes of such $M_n(A)$ is $\sum_{\ell=3}^{n-2} [\frac{\ell}{2}]$.

Proof: We associate with the pair (y, x) the Bott matrix (14) where $y = n - x$ and x are the numbers of zero entries in the $(k-1)$-th row and k-th row respectively of the right-upper block matrix. Here $1 \leq x \leq \ell - 1$. Because of move I, we may assume that $x \leq \ell - x$. So $1 \leq x \leq [\frac{\ell}{2}]$. For a fixed numbers ℓ and x, it is easy to check that the fixed point sets of $((\mathbb{Z}_2)^2, T^\ell)$ corresponding to (14) are 2^x components $T^{\ell-x}$ and $2^{\ell-x}$ components T^x.

For a fixed number ℓ, suppose that Bott matrices A_1 and A_2 correspond to the pairs (y_1, x_1) and (y_2, x_2) respectively. If $x_1 \neq x_2$, then by Proposition 1, $M_n(A_1)$ is not diffeomorphic to $M_n(A_2)$.

Therefore for a fixed number ℓ, there are $[\frac{\ell}{2}]$ diffeomorphism classes of such RBM_ns. This implies the lemma. ■

Lemma 2: Let $M_n(A)$ be a RBM_n $(n \geq 5)$ corresponding to the Bott matrix

$$A = \left(\begin{array}{c|ccccccccc} & & & & & 0 & & & & \\ O_k & 0 & \ldots & 0 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ \hline 0 & & & & & O_\ell & & & & \end{array} \right) \quad (\ell \geq 3). \tag{15}$$

Then the number of diffeomorphism classes of such $M_n(A)$ is

$$\sum_{\ell=3}^{n-2} \sum_{x=1}^{[\frac{\ell}{3}]} \left(\left[\frac{\ell - x}{2}\right] - (x-1) \right).$$

Proof: We associate with the pair (t, x) the Bott matrix (15) where x is the number of zero entries in the k-th row of the right-upper block matrix and $t (\neq 0)$ is the number of columns having two non zero entries. Because of move I, we may assume that $1 \leq x \leq t \leq \ell - x - t$ and $x \leq [\frac{\ell}{3}]$. So $1 \leq x \leq t \leq [\frac{\ell-x}{2}]$. For fixed numbers ℓ, x and t, it is easy to check that the fixed point sets of $((\mathbb{Z}_2)^2, T^\ell)$ corresponding to (15) are 2^{t+x} components $T^{\ell-x-t}$ and $2^{\ell-x}$ components T^x.

For a fixed number ℓ, suppose that Bott matrices A_1 and A_2 correspond to the pairs (t_1, x_1) and (t_2, x_2) respectively. If $t_1 \neq t_2$ or(and) $x_1 \neq x_2$, then by Proposition 1, A_1 is not equivalent to A_2.

Therefore for fixed numbers ℓ and x, there are $\left[\frac{\ell-x}{2}\right] - (x - 1)$ diffeomorphism classes of such RBM_ns. Hence there are

$$\sum_{\ell=3}^{n-2} \sum_{x=1}^{[\frac{\ell}{3}]} \left(\left[\frac{\ell - x}{2}\right] - (x-1) \right)$$

diffeomorphism classes of such $M_n(A)$ corresponding to Bott matrices as in (15). ■

Since the fixed point sets of $((\mathbb{Z}_2)^2, T^\ell)$ corresponding to Bott matrices (14) and (15) are different, the corresponding real Bott manifolds are not diffeomorphics.

$$A = \left(\begin{array}{c|cccccccc} & & & & 0 & & & & \\ & & & 0 & & & 1 \ \ldots \ 1 & & 1 \ \ldots \ 1 \\ O_k & & & & & \cdot & & & \\ & & & & \cdot & & & & \\ & & & \cdot & & & 0 & & \\ & & 1 \ \ldots \ 1 & & & & & & \\ & 1 \ \ldots \ 1 & & & & & & & \\ \hline 0 & & & & O_\ell & & & & \end{array}\right) \tag{12}$$

$$(\ell \geq 3) \tag{13}$$

Remark 10: It is hard task algebraically to determine the number of n-dimensional $M_n(A) = T^k \times_{(\mathbb{Z}_2)^s} T^\ell$ for $3 \leq s \leq min\{n-\ell, \ell\}$. However we shall consider a special type in (12).

We associate with $(\ell_1, \ell_2, \ldots, \ell_{s-1}, \ell_s)$ the Bott matrix (12) where $\ell_1 = \ell - \sum_{i=2}^{s} \ell_i, \ell_2, \ell_3, \ldots, \ell_{s-1}, \ell_s$ are the number of nonzero entries at k-row, $(k-1)$-row, $(k-2)$-row, $\ldots$, $(k-(s-2))$-row, $(k-(s-1))$-row respectively in the right-upper block matrix. As in the arguments in the proof of Lemmas 1, 2 above, in order to obtain the diffeomorphism classes $RBM\ M(A)$, we may assume that $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_{s-1} \geq \ell_s \geq 1$ and $1 \leq \ell_s \leq [\frac{\ell}{s}]$. For any ℓ_s we determine the values of ℓ_{s-1}, namely $\ell_s \leq \ell_{s-1} \leq [\frac{\ell-\ell_s}{s-1}]$. For any ℓ_{s-1}, similarly we can determine that $\ell_{s-1} \leq \ell_{s-2} \leq [\frac{\ell-\ell_s-\ell_{s-1}}{s-2}]$. Repeating the previous argument, we obtain that

$$\ell_{t+1} \leq \ell_t \leq [\frac{\ell - \sum_{i=t+1}^{s} \ell_i}{t}], \quad t = 2, \ldots, s-2, s-1.$$

It is easy to check that for fixed natural numbers ℓ_t, $t = 3, 4, \ldots, s-1, s$ and ℓ, there are $\left[\frac{\ell - \sum_{i=3}^{s} \ell_i}{2}\right] - (\ell_3 - 1)$ diffeomorphism classes of $RBM\ \ M(A)$. Hence for fixed numbers ℓ and s, there are

$$N_\ell^s = \sum_{\ell_s=1}^{[\frac{\ell}{s}]} \sum_{\ell_{s-1}=\ell_s}^{[\frac{\ell-\ell_s}{s-1}]} \cdots \sum_{\ell_t=\ell_{t+1}}^{\left[\frac{\ell-\sum_{i=t+1}^{s}\ell_i}{t}\right]} \cdots \sum_{\ell_3=\ell_4}^{\left[\frac{\ell-\sum_{i=4}^{s}\ell_i}{3}\right]} [\frac{\ell - \sum_{i=3}^{s} \ell_i}{2}] - (\ell_3 - 1)$$

diffeomorphism classes of $M(A)$ for $3 \leq s \leq min\{n-\ell, \ell\}$.

Hence the number of diffeomorphism classes of $RBM\ M(A)$ is

$$\sum_{\ell=3}^{n-3} \sum_{s=3}^{min\{n-\ell,\ell\}} N_\ell^s.$$

Let N_n be the number of diffeomorphism classes of RBM_ns.

Choi[7] classified RBM_ns corresponding to the following Bott matrices. He considers $\ell \times \ell$ Bott matrices A_ℓ of rank $\ell - 1$ all of whose diagonals are 0. Then for such each A_ℓ, $(i, i+1)$-entry must be 1 for $i = 1, \ldots, \ell-1$. Masuda[8] proved that for such matrices, there are $2^{(\ell-2)(\ell-3)/2}$ diffeomorphism classes of ℓ-dimensional real Bott manifolds.

Next Choi considers an $n \times n$ Bott matrix A such that the rank of submatrix consisting of the first ℓ columns is $\ell - 1$ and the last $n - \ell$ columns are zero vectors (i.e, $A = \begin{pmatrix} A_\ell & 0 \\ 0 & 0 \end{pmatrix}$). By move **I**, the Bott matrix A is equivalent to

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_\ell \end{pmatrix}. \tag{16}$$

By using the result of Masuda above, Choi [7] obtained that the number of diffeomorphism classes of RBM_ns corresponding to Bott matrices (16) for $\ell = 2, \ldots, n$ is $\sum_{\ell=2}^{n} 2^{(\ell-2)(\ell-3)/2}$.

Masuda [8] found that

$$2^{(n-2)(n-3)/2} \leq N_n,$$

by considering the Bott matrices A_ℓ above. Then, Choi [7] improved the Masuda's result where he considers Bott matrices (16).

Theorem 3 ([7]): $2^{(n-2)(n-3)/2} < \sum_{\ell=2}^{n} 2^{(\ell-2)(\ell-3)/2} \leq N_n$.

By using Propositions 7, 2, Theorem 3, Remarks 3, 4, 5, 6, 7, 8, 9, 10, we obtain an improvement of the previous results about N_n.

Theorem 4: For $n \geq 4$,

$$\begin{aligned} N_n \geq\ & 8n + \sum_{\ell=2}^{n} 2^{(\ell-2)(\ell-3)/2} + \sum_{\ell=2}^{n-2} ([\frac{\ell}{2}] + 1) + \\ & \sum_{k=1}^{n-5} \sum_{t=k+2}^{n-3} ([\frac{n-t}{2}] + 1) \sum_{\ell=5}^{n} \sum_{m=2}^{\ell-3} (m+1) 2^{\ell-m-3} + \\ & \sum_{\ell=3}^{n-2} ([\frac{\ell}{2}] + \sum_{x=1}^{[\frac{\ell}{3}]} ([\frac{\ell-x}{2}] - (x-1))) + \\ & \sum_{\ell=3}^{n-3} \sum_{s=3}^{min\{n-\ell,\ell\}} N_\ell^s - 26 \end{aligned}$$

with

$$N_\ell^s = \sum_{\ell_s=1}^{[\frac{\ell}{s}]} \sum_{\ell_{s-1}=\ell_s}^{[\frac{\ell-\ell_s}{s-1}]} \cdots \sum_{\ell_t=\ell_{t+1}}^{\left[\frac{\ell-\sum_{i=t+1}^{s}\ell_i}{t}\right]} \cdots \sum_{\ell_3=\ell_4}^{\left[\frac{\ell-\sum_{i=4}^{s}\ell_i}{3}\right]} [\frac{\ell - \sum_{i=3}^{s} \ell_i}{2}] - (\ell_3 - 1).$$

We assume that if $u < u_0$ in a summation $\sum_{\ell=u_0}^{u}$, the value of such summation is equal to zero.

REFERENCES

[1] Y. Kamishima and M. Masuda, "Cohomological rigidity of real bott manifolds," *Algebraic & Geometric Topology*, vol. 9, no. 4, pp. 2479–2502, 2009.

[2] Y. Kamishima and A. Nazra, "Seifert fibred structure and rigidity on real bott towers," *Contemporary Mathematics*, vol. 501, p. 103, 2009.

[3] J. A. Wolf, *Spaces of constant curvature*. American Mathematical Soc., 1972, vol. 372.

[4] A. Nazra, "Real bott tower," Master's thesis, Tokyo Metropolitan University, 2008.

[5] ——, "Diffeomorphism type of real bott towers (geometry of transformation groups and related topics)," 2008.

[6] ——, "Diffeomorphism classes of real bott manifolds," *Tokyo Journal of Mathematics*, vol. 34, no. 1, pp. 229–260, 2011.

[7] S. Choi, "Counting on real bott manifolds," *RIMS Kokyuroku*, vol. 1670, pp. 69–71, 2009.

[8] M. Masuda, "Classification of real bott manifolds," *arXiv preprint arXiv:0809.2178*, 2008.