Properties of Generalised Lattice Ordered Groups

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***Abstract*—A partially ordered group (po-group) is said to be a generalised lattice ordered group (gl-group) if the underlying poset is a generalised lattice. This paper is a study of some properties of finite subsets of a generalised lattice ordered group (gl-group). Finally obtained a lattice ordered group (l-group) from the given interally closed gl-group and concluded that every integrally closed gl-group is distributive.**

***Index Terms*—Poset, lattice, po-group, l-group.**

I. Introduction

Murty and Swamy [1] introduced the concept of a generalised lattice and Kishore [2], [3], developed the theory of generalised lattices. The theory of lattice ordered groups (l-groups) is well known from the books [4], [5], [6]. The concept of generalised lattice ordered groups (gl-group) introduced and developed by Kishore [7], [8], [9]. This paper is a study of some properties of finite subsets of a gl-group. In this paper, Section II contains preliminaries which are taken from the references [7], [8]. In Section III, we proved some properties of a gl-group with respect to the elements of the gl-group. Section IV discussed some properties of finite subsets of gl-groups. In Section V, we obtained an l-group from a given integrally closed gl-group and finally concluded that every integrally closed gl-group is distributive.

II. Preliminaries

The definitions of partially ordered group (po-group), totally ordered group (o-group), lattice ordered group (l-group), directed group are well known from the books [4], [5], [6]. The additive identity element of a po-group is denoted by 0. $G^+ = \{x \in G \mid x \geq 0\}$ which is called positive cone of a po-group G. A po-group G is said to be integrally closed if for any $a, b \in G;\ na \leq b$ for all $n \in \mathbb{N}$ implies $a \leq 0$. A po-group G is said to be semiclosed if for any $x \in G,\ n \in \mathbb{N};\ nx \geq 0$ implies $x \geq 0$.

The concepts of generalised lattice, subgeneralised lattice, distributive poset are known from [2], [3], [1]. For any finite subset A of a poset P, define $L(A) = \{x \in P \mid x \leq a$ for all $a \in A\}$, then the set $\mathcal{L}(P) = \{L(A) \mid A$ is a finite subset of $P\}$ is a semi lattice under the set inclusion. If a poset P is a generalised lattice then $(\mathcal{L}(P), \subseteq)$ is a lattice. A generalised lattice P is distributive if and only if $\mathcal{L}(P)$ is distributive. The dual concepts are also true for $U(A)$ and $\mathcal{U}(P)$.

Definition 1 ([7]): A system $(G,\ +\ \leq)$ is called a *gl-group* (generalised lattice ordered group) if (i) $(G,\ \leq)$ is a generalised lattice, (ii) $(G,\ +)$ is a group and (iii) every group translation $x \to a + x + b$ on G is isotone, i.e., $x \leq y$ implies $a + x + b \leq a + y + b$ for all $a, b \in G$.

Here onwards through out this paper G denotes a gl-group unless specified otherwise. Let X, Y, A and B be subsets of G. Define $X \leq Y$ if $x \leq y$ for all $x \in X, y \in Y$. Define $A + X = \{a + x \mid a \in A, x \in X\}$. In particular if $A = \{a\}$ then $A + X = a + X$. Observe that the following conditions are equivalent: (iii) of Definition 1, (iii) $':\ X \leq Y$ implies $a + X + b \leq a + Y + b$ for all $a, b \in G$ and (iii) $'':\ X \leq Y$ implies $A + X + B \leq A + Y + B$.

Theorem 1 ([7]): For any $x, y, a, b \in G$, we have the following properties: (iv) $a + mu\{x, y\} + b = mu\{a + x + b, a + y + b\},\ a + ML\{x, y\} + b = ML\{a + x + b, a + y + b\}$. (v) $ML\{a, b\} = \{0\}$ and $ML\{a, c\} = \{0\} \implies ML\{a, b + c\} = \{0\},\ mu\{a, b\} = \{0\}$ and $mu\{a, c\} = \{0\} \implies mu\{a, b + c\} = \{0\}$. (vi) $ML\{x, y\} = -mu\{-x, -y\},\ mu\{x, y\} = -ML\{-x, -y\}$. (vii) $a - mu\{x, y\} + b = ML\{a - x + b, a - y + b\},\ a - ML\{x, y\} + b = mu\{a - x + b, a - y + b\}$. (viii) $x - mu\{x, y\} + y = ML\{x, y\},\ x - ML\{x, y\} + y = mu\{x, y\}$.

Definition 2 ([8]): For any $x \in G$, define $|x| = mu\{x, -x\},\ x^+ = mu\{x, 0\}$ and $x^- = mu\{-x, 0\}$.

Note that x^- may also be defined as $x^- = ML\{x, 0\}$, but both differ only in negative sign of a set, that is one is negative set of the other. In this paper, we consider as given in Definition 2.

Theorem 2 ([8]): For any $x \in G$, we have (ix) $x^+ = x + x^-,\ x^- = -x + x^+,\ (-x)^+ = x^-,\ (-x)^- = x^+$. (x) If G is semiclosed then $|x| = |-x| \geq 0,\ |x| = \{0\} \Leftrightarrow x = 0,\ L(|x|) = L(x^+) \vee L(x^-)$. (xi) If G is distributive and semiclosed then $L(x^+) \cap L(x^-) = L(0)$.

III. Properties of a gl-group w.r.t. its elements

In this section, we prove some properties of a gl-group with respect to elements of the gl-group.

Theorem 3: If G is distributive and semiclosed then for any $x, y \in G$, we have $ML((x - ML\{x, y\}) \cup (y - ML\{x, y\})) = \{0\}$.

Proof: Consider $L((x - ML\{x, y\}) \cup (y - ML\{x, y\})) = L(mu\{0, x - y\} \cup mu\{y - x, 0\}) = (L(0) \vee L(x - y)) \cap (L(y - x) \vee L(0)) = L(0) \vee (L(x - y) \cap L(y - x))$ (since $\mathcal{L}(G)$ is distributive) $= L(0)$ (since G is semiclosed). ■

Theorem 4: For any $x, y \in G$, we have $ML(|x|) \leq y$ if and only if $-y \leq x \leq y$.

Proof: Suppose $ML(|x|) \leq y$. Then $L(x) \vee L(-x) = L(mu\{x, -x\}) = L(|x|) \subseteq L(y)$ and therefore $L(x), L(-x) \subseteq L(y)$; that is $-y \leq x \leq y$. Conversely, suppose $-y \leq x \leq y$. Then $y \in U(s)$ for some $s \in mu\{x, -x\}$ and therefore $ML(|x|) \subseteq L(|x|) \subseteq L(s) \subseteq L(y)$; that is $ML(|x|) \leq y$. ■

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Theorem 5: If G is distributive and semiclosed, then for any $x, y \in G$, the following statements are equivalent: (i) $ML\{x, y\} = \{0\}$ (ii) $mu\{x, y\} = \{x + y\}$ (iii) $(x - y)^+ = \{x\}$ and $(x - y)^- = \{y\}$.

Proof: (i) if and only if (ii) is clear by the Theorem 1. Now to prove (iii) if and only if (i): Suppose (iii). Then since G is distributive and semiclosed, we have $L(\{x, y\}) = L((x-y)^+) \cap L((x - y)^-) = L(0)$; that is $ML\{x, y\} = \{0\}$. The converse is clear by the Theorem 1. ∎

Theorem 6: If G^+ is a subgeneralised meet semilattice of G, then for any $x, y, z \in G^+$, we have (i) $x \leq y + z$ implies $x = y_1 + z_1$ for some $0 \leq z_1 \leq z$ and $0 \leq y_1 \leq y$. (ii) $ML\{x, y + z\} \leq mu(ML\{x, y\} + ML\{x, z\})$.

Proof: (i) Suppose $x \leq y + z$. Then $z \in U(z_1)$ for some $z_1 \in mu\{0, -y+x\} = -ML\{x, y\}+x$. Therefore, $0 \leq z_1 \leq z$ and $x = y_1 + z_1$ for some $y_1 \in ML\{x, y\}$. Since G^+ is a subgeneralised meet semilattice of G, we have $ML\{x, y\} \subseteq G^+$; this gives $0 \leq y_1 \leq y$. (ii) Let $s \in ML\{x, y + z\}$. Then by (i) $s = y_1 + z_1$ for some $0 \leq z_1 \leq z$ and $0 \leq y_1 \leq y$. Since $y_1, z_1 \leq s \leq x$; we can get $y_1 \leq y_2$ and $z_1 \leq z_2$ for some $y_2 \in ML\{x, y\}, z_2 \in ML\{x, z\}$. Therefore, $U(ML\{x, y\} + ML\{x, z\}) \subseteq U(y_2 + z_2) \subseteq U(s)$ for all $s \in ML\{x, y + z\}$, this implies $U(ML\{x, y\} + ML\{x, z\}) \subseteq U(ML\{x, y + z\})$ and hence the result. ∎

Corollary 1: If G^+ be a subgeneralised lattice of G, then for any $x, y, z \in G^+$ we have $ML\{x, y\} = ML\{x, z\} = \{0\}$ implies $ML\{x, y + z\} = \{0\}$.

IV. Properties of a gl-group w.r.t. its finite subsets

In this section, we prove some properties of a gl-group with respect to finite subsets of the gl-group.

Theorem 7: For any finite subset X of G and $a, b \in G$, we have the following: (i) $a + mu(X) = mu(a + X),\ mu(X) + b = mu(X + b),\ a + mu(X) + b = mu(a + X + b)$ and (ii) $a + ML(X) = ML(a + X),\ ML(X) + b = ML(X + b),\ a + ML(X) + b = ML(a + X + b)$.

Proof: For any $s \in mu(X)$, we have $a + x \leq a + s$ for all $x \in X$. Therefore, $a + s \in U(a + X)$ for all $s \in mu(X)$, and this implies $\bigcup_{s\in mu(X)} U(a + s) \subseteq U(a + X)$. On the other hand let $t \in U(a + X)$, then there exists $s \in mu(X)$ such that $-a + t \in U(s)$ and so that $t \in \bigcup_{s\in mu(X)} U(a + s)$. Therefore, $U(a + X) \subseteq \bigcup_{s\in mu(X)} U(a + s)$. Hence, $U(a + X) = \bigcup_{s\in mu(X)} U(a + s)$ and then we get $mu(a + X) = a + mu(X)$. Similarly, we can prove the remaining. ∎

Theorem 8: For any finite subsets A, B, C of G, we have the following: (i) $ML(A \cup B) = \{0\},\ ML(A \cup C) = \{0\}$ implies $ML(A \cup (B + C)) = \{0\}$ and (ii) $mu(A \cup B) = \{0\},\ mu(A \cup C) = \{0\}$ implies $mu(A \cup (B + C)) = \{0\}$.

Proof: (i) Suppose $ML(A \cup B) = \{0\},\ ML(A \cup C) = \{0\}$. Then clearly $0 \in L(A \cup (B + C))$. Now let $p \in L(A \cup (B+C))$, then $p \in L((A+A)\cup(B+A)\cup(A+C)\cup(B+C)) = L((A\cup B)+(A\cup C))$, later for any $y \in A\cup C$ we have $p-y \in L(A \cup B) = L(0)$, and this implies $p \in L(A \cup C) = L(0)$. Therefore $ML(A \cup (B + C)) = \{0\}$. Similarly, we can prove (ii). ∎

Theorem 9: For any finite subset X of G and $a, b \in G$, we have the following: (i) $ML(X) = -mu(-X)$ (ii) $mu(X) = -ML(-X)$ (iii) $a - mu(X) + b = ML(a - X + b)$ and (iv) $a - ML(X) + b = mu(a - X + b)$.

Theorem 10: Let G be a po-group. Then G is a gl-group if and only if $mu(X \cup \{0\})$ (or $ML(X \cup \{0\})$) exists for any finite subset X of G.

Proof: Suppose G is a gl-group. Then for any finite subset X of G, since $X \cup \{0\}$ is also a finite subset of G, we have $mu(X \cup \{0\})$ and $ML(X \cup \{0\})$ are finite subsets of G. Conversely, suppose the condition. Let A be a finite subset of G. Then for any $x \in A$, by Theorem 7, we have $-ML(A) = mu(-A) = mu(\{0\} \cup (-A + x)) + \{-x\}$ is a finite subset of G. Therefore $ML(A)$ is a finite subset of G and clearly $mu(A)$ is a finite subset of G. ∎

Theorem 11: Let G be a po-group. Then G is a gl-group if and only if G^+ is subgeneralised join semilattice of G and G^+ generates G (i.e., $G = G^+ - G^+$).

Proof: Suppose G is a gl-group. Then for any finite subset A of G^+, since $A \subseteq G$ and $A \geq 0$, we have $mu(A) \subseteq G$ and $mu(A) \geq 0$, that is $mu(A) \subseteq G^+$. Therefore, G^+ is a subgeneralised join semilattice of G. Since every generalised lattice is directed, by theorem 2.1.2(c) of [6] we have G^+ generates G (i.e., $G = G^+ - G^+$). Conversely, suppose the condition. Let $x \in G$, then $x = a - b$ for some $a, b \in G^+$. Observe that an element g is a minimal upper bound of $\{a, b\}$ in G^+ if and only if g is a minimal upper bound of $\{a, b\}$ in G. Since G^+ is subgeneralised join semilattice of G and $a, b \in G^+$, the set of minimal upper bounds of $\{a, b\}$ in G^+ is a finite subset of G^+. Then the set of minimal upper bounds of $\{a, b\}$ in G is a finite subset of G. Now, $mu\{x, 0\} = mu\{a-b, 0\} = mu\{a, b\} - b$ is a finite subset of G. Therefore, $mu\{x, 0\}$ is a finite subset of G for all $x \in G$. Hence, $mu(X \cup \{0\})$ exists for any finite subset X of G. ∎

V. Integrally closed gl-groups

In this section, we obtain an l-group from a given integrally closed gl-group and concluded that every integrally closed gl-group is distributive.

Recall from [6] the following results. Let G be a directed po-group and $\mathcal{P}(G)$ be the powerset of G. Then observe that the map $\sigma : \mathcal{P}(G) \to \mathcal{P}(G)$ defined by $\sigma(X) = LU(X)$ is a closure operation on G. Moreover for each $X \in \mathcal{P}(G),\ \sigma(X) = \bigcap_{A\in\mathcal{A}} A$ where $\mathcal{A} = \{A \mid X \subseteq A \subseteq G$ and $\sigma(A) = A\}$.

In the following result, we obtain an l-monoid (l-group) from a given gl-group (integrally closed gl-group).

Theorem 12: Let G be a gl-group. Then $(\mathcal{L}(G), \oplus, \subseteq)$ is an l-monoid under the operation $\oplus$ defined by $L(A) \oplus L(B) = L(mu(ML(A) + ML(B)))$ for any $L(A), L(B) \in \mathcal{L}(G)$. Moreover, if G is integrally closed then $\mathcal{L}(G)$ is an l-group.

Proof: Closure: Since $ML(A), ML(B)$ are finite subsets of G, $mu(ML(A) + ML(B))$ is also a finite subset of G. Identity: $L(0)$ is the identity element. Associative: Let $L(A), L(B), L(C) \in \mathcal{L}(G)$. Consider $(L(A) \oplus L(B)) \oplus L(C) = L(D) \oplus L(C)$ (where $D = mu(ML(A) + ML(B))$) $= LU(X) = X^*$ (where $X = ML(D) + ML(C)$) $= \bigcap_{Q\in\mathcal{A}} Q$ where $\mathcal{A} = \{Q \mid X \subseteq Q \subseteq G, Q^* = Q\}$. Consider $L(A) \oplus (L(B) \oplus L(C)) = L(A) \oplus L(E)$ (where

$E = mu(ML(B) + ML(C))) = LU(Y) = Y^*$ (where $Y = ML(A) + ML(E)) = \bigcap_{P \in \mathcal{B}} P$ where $\mathcal{B} = \{P \mid Y \subseteq P \subseteq G, P^* = P\}$. To show that $\mathcal{A} = \mathcal{B}$: Let $Q \in \mathcal{A}$. Then for any $z \in ML(C)$ we have $ML(D) + z \subseteq X \subseteq Q$, this implies $L(D) = LUL(D) = LU(ML(D)) \subseteq (Q - z)^* = Q - z$ for all $z \in ML(C)$, later since $ML(A) + ML(B) \subseteq L(D)$ we get $ML(A) + (ML(B) + ML(C)) = (ML(A) + ML(B)) + ML(C) \subseteq L(D) + ML(C) \subseteq Q$. Then for any $z \in ML(A)$ we have $z + (ML(B) + ML(C)) \subseteq Q$, this implies $L(E) = LU(ML(B) + ML(C)) \subseteq (-z + Q)^* = -z + Q$ for all $z \in ML(A)$, later since $Y \subseteq ML(A) + L(E) \subseteq Q$ we have $Q \in \mathcal{B}$. Therefore $\mathcal{A} \subseteq \mathcal{B}$, similarly we can prove $\mathcal{B} \subseteq \mathcal{A}$. Hence $(\mathcal{L}(G), \oplus)$ is a monoid. Translation order preserving: Let $L(A), L(B), L(C) \in \mathcal{L}(G)$ and suppose $L(A) \subseteq L(B)$. Then since $U(ML(B) + ML(C)) \subseteq U(ML(A) + ML(C))$ we have $L(A) \oplus L(C) \subseteq L(B) \oplus L(C)$ and similarly $L(C) \oplus L(A) \subseteq L(C) \oplus L(A)$. Therefore $(\mathcal{L}(G), \oplus, \subseteq)$ is an l-monoid. Now suppose G is integrally closed. Inverse: Let $L(A) \in \mathcal{L}(G),\ X = ML(A)$ and $Y = ML(mu(-A))$. Then clearly $L(A) \oplus L(mu(-A)) = LU(X + Y) \subseteq L(0)$. On the other hand let $a \in U(X + Y)$. To show that $na \in U(X + Y)$ for all positive integers n: We prove this by induction on n. Assume that it is true for $n = k$, that is $ka \in U(X + Y)$. Then for any $x \in X, y \in Y$ we have $ka \geq x + y$, this implies $y - ka \leq t$ for some $t \in Y$, later since $U(X + Y) \subseteq U(x + t) \subseteq U(x + y - ka)$ we have $a \geq x + y - ka$, that is $(k + 1)a \in U(X + Y)$. Therefore the result follows by induction. Now since G is integrally closed we get $0 \leq a$ for all $a \in U(X + Y)$, this implies $L(0) \subseteq LU(X + Y) = L(A) \oplus L(mu(-A))$. Therefore $L(mu(-A))$ is the inverse of $L(A)$. Hence $(\mathcal{L}(G), \oplus, \subseteq)$ is an l-group. ■

Recall that every l-group is distributive w.r.t. the lattice operations.

Corollary 2: Every integrally closed gl-group is distributive.

Corollary 3: Every integrally closed gl-group is semiclosed.

Theorem 13: Let G be an integrally closed gl-group. Then for any $x, y \in G$, we have $ML(|x - y|) = ML(mu(ML(mu\{x, y\}) - mu(ML\{x, y\})))$. In particular, if $y = 0$ then $ML(|x|) = ML(mu(ML(x^+) + ML(x^-)))$.

Proof: Since $L(0) \subseteq L(|x - y|) = L(x - y) \vee L(y - x)$ and by Theorem 12 we have $L(|x - y|) = L(0) \vee L(x - y) \vee L(y - x) \vee L(0) = (L(x) \oplus L(-x)) \vee (L(x) \oplus L(-y)) \vee (L(y) \oplus L(-x)) \vee (L(y) \oplus L(-y)) = (L(x) \oplus (L(-x) \vee L(-y))) \vee (L(y) \oplus (L(-x) \vee L(-y))) = (L(x) \vee L(y)) \oplus (L(-x) \vee L(-y)) = L(mu\{x, y\}) \oplus L(mu\{-x, -y\}) = L(mu(ML(mu\{x, y\}) - mu(ML\{x, y\})))$. ■

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