On the Reciprocal Sums of Generalized Fibonacci-Like Sequence

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***Abstract*—The Fibonacci and Lucas sequences have been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. One of them is defined by the relation $B_n = B_{n-1} + B_{n-2} \quad n \geq 2$ with the initial condition $B_0 = 2s,\ B_1 = s+1$ where $s \in \mathbb{Z}$. In this paper, we consider the reciprocal sums of B_n and B_n^2, with an established result that also involve B_n.**

***Index Terms*—Reciprocal sums, generalized Fibonacci-like sequence.**

I. Introduction

MANY author have already generalize a well known Fibonacci and Lucas sequence either by changing its initial condition or the recurrence relation. One of that generalization is called the Generalized Fibonacci-Like sequence [1]. The Generalized Fibonacci-Like sequence [1] associated with Fibonacci and Lucas sequences $\{B_n\}$ is defined by the recurrence relation

$$B_n = B_{n-1} + B_{n-2} \quad n \geq 2$$

with the initial condition $B_0 = 2s,\ B_1 = s+1$ where $s \in \mathbb{Z}$. The few terms of this sequence are as following

$$2s, s+1, 3s+1, 4s+2, 7s+3, \ldots$$

The initial condition B_0 and B_1 can be seen as the sum of Fibonacci and Lucas sequence respectively

$$B_0 = F_0 + sL_0 \qquad B_1 = F_1 + sL_1$$

Thus, the relation between Fibonacci-Lucas sequence with Generalized Fibonacci-Like sequence can be written as

$$B_n = F_n + sL_n \quad (n \geq 0)$$

If $s = 0$, then B_n become a usual Fibonacci sequence. If $s = 1$, then B_n become a usual Pell-Lucas sequence. In this article, we discuss the results when $s = 2$. The few terms of this sequence are

$$4, 3, 7, 10, 17, 27, 44, \ldots$$

The reciprocal sum of Fibonacci numbers was first investigated by Ohtsuka and Nakamura [2]. Some related result for other sequences also have been founded by several authors [3], [4], [5], [6], [7], [8], [9]. In this article, we discuss the infinite reciprocal sums of generalized Fibonacci-Like sequence and additionally the infinite reciprocal sums of square of generalized Fibonacci-Like sequence when $s = 2$.

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II. Preliminaries

Various properties and identities of Fibonacci and Lucas sequences have been studied by many authors [10], [11]. We give some identities on Generalized Fibonacci-Like sequence in order to help prove our main results.

Lemma 1: For $n \geq 1$, we have

1) $B_{n-1}B_{n+3} = B_{n+1}^2 + (-1)^{n+1}(5s^2 - 1)$
2) $B_n B_{n+2} = B_{n+1}^2 - (-1)^{n+1}(5s^2 - 1)$
3) $B_n^2 - B_{n-1}B_{n+1} = (-1)^n(5s^2 - 1)$

Proof: Observe that

$$\begin{aligned} B_{n+1}^2 &= (F_{n+1} + sL_{n+1})^2 \\ &= F_{n+1}^2 + 2sF_{n+1}L_{n+1} + s^2L_{n+1}^2 \\ &= F_{n+1}^2 + 2sF_{2n+2} + s^2\left(L_{2n+2} + 2(-1)^{n+1}\right) \end{aligned}$$

1) We have

$$\begin{aligned} B_{n-1}B_{n+3} &= (F_{n-1} + sL_{n-1})(F_{n+3} + sL_{n+3}) \\ &= F_{n-1}F_{n+3} + s(F_{n-1}L_{n+3} + L_{n-1}F_{n+3}) \\ &\quad + s^2L_{n-1}L_{n+3} \\ &= F_{n+1}^2 + (-1)^n + s(2F_{2n+2}) + s^2(L_{2n+2} \\ &\quad + 7(-1)^{n+1}) \\ &= B_{n+1}^2 + (-1)^{n+1}(5s^2 - 1) \end{aligned}$$

Thus (1) is proved and (2) is proved in a similar way.

3) From (2), we get

$$\begin{aligned} B_n^2 - B_{n-1}B_{n+1} &= B_n^2 - (B_n^2 - (-1)^n(5s^2 - 1)) \\ &= (-1)^n(5s^2 - 1) \end{aligned}$$

The proof is complete. ■

III. Results and Discussion

There are two main results in our studies, the first one is as following.

Theorem 1: If $s = 2$, then

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k} \right)^{-1} \right\rfloor = \begin{cases} B_{n-2}, & \text{if n is odd and } n \geq 3 \\ B_{n-2} - 1, & \text{if n is even and } n \geq 4 \end{cases}$$

To prove the first theorem, we use the following two lemmas.

Lemma 2: For any $s \in \mathbb{Z}^+$,

(a) $\displaystyle\sum_{k=n}^{\infty} \frac{1}{B_k} < \frac{1}{B_{n-2}}$, if n is odd and $n \geq 3$.

(b) $\displaystyle\sum_{k=n}^{\infty} \frac{1}{B_k} > \frac{1}{B_{n-2}}$, if n is even and $n \geq 2$.

Proof: For $n \geq 0$, observe that

$$\begin{aligned} \frac{1}{B_n} - \frac{2}{B_{n+2}} - \frac{1}{B_{n+3}} &= \frac{B_{n-1}}{B_n B_{n+2}} - \frac{1}{B_{n+3}} \\ &= \frac{B_{n-1}B_{n+3} - B_n B_{n+2}}{B_n B_{n+2} B_{n+3}} \\ &= \frac{(-1)^n(2-10s^2)}{B_n B_{n+2} B_{n+3}} \end{aligned}$$

(a) If n is odd with $n \geq 1$, then

$$\frac{1}{B_n} - \frac{2}{B_{n+2}} - \frac{1}{B_{n+3}} = \frac{(-1)^n(2-10s^2)}{B_n B_{n+2} B_{n+3}} > 0$$

Therefore,

$$\frac{1}{B_n} > \frac{1}{B_{n+2}} + \frac{1}{B_{n+2}} + \frac{1}{B_{n+3}} \tag{1}$$

By applying inequality (1) repeatedly for $n \geq 3$, we have

$$\begin{aligned} \frac{1}{B_{n-2}} &> \frac{1}{B_n} + \frac{1}{B_n} + \frac{1}{B_{n+1}} \\ &> \frac{1}{B_n} + \frac{1}{B_{n+1}} + \left(\frac{1}{B_{n+2}} + \frac{1}{B_{n+2}} + \frac{1}{B_{n+3}}\right) \\ &> \dots \\ &> \frac{1}{B_n} + \frac{1}{B_{n+1}} + \frac{1}{B_{n+2}} + \frac{1}{B_{n+3}} + \dots \end{aligned}$$

Thus, we obtain

$$\sum_{k=n}^{\infty} \frac{1}{B_k} < \frac{1}{B_{n-2}}$$

(b) In a similar way, if n is even with $n \geq 2$, then we will get

$$\sum_{k=n}^{\infty} \frac{1}{B_k} > \frac{1}{B_{n-2}}$$

The proof is complete. ∎

Lemma 3: For $s = 2$, we have

(a) $\displaystyle\sum_{k=n}^{\infty} \frac{1}{B_k} > \frac{1}{B_{n-2}+1}$, with $n > 3$.

(b) $\displaystyle\sum_{k=n}^{\infty} \frac{1}{B_k} < \frac{1}{B_{n-2}-1}$, with $n \geq 3$.

Proof:

(a) Using identities on Generalized Fibonacci-Like sequence, we have

$$\begin{aligned} &\frac{1}{B_n+1} - \frac{1}{B_{n+2}} - \frac{1}{B_{n+3}} - \frac{1}{B_{n+2}+1} \\ &= \frac{B_{n+2} - B_n}{(B_n+1)(B_{n+2}+1)} - \frac{B_{n+2} + B_{n+3}}{B_{n+2}B_{n+3}} \\ &= \frac{B_{n+1}}{(B_n+1)(B_{n+2}+1)} - \frac{B_{n+4}}{B_{n+2}B_{n+3}} \\ &= \frac{-\left(B_{n+2}^2 + B_{n+3}^2 + B_{n+4}\right)}{B_{n+2}B_{n+3}(B_n+1)(B_{n+2}+1)} + \\ &\quad + \frac{\left(38(-1)^{n+1}(B_{n+2}+1)\right)}{B_{n+2}B_{n+3}(B_n+1)(B_{n+2}+1)} \end{aligned} \tag{2}$$

If n is even, then the right-hand side of identity (2) will be negative.

If n is odd , then the right-hand side of identity (2) will also be negative, except for $n = 1$ because

$$-\left(B_3^2 + B_4^2 + B_5 - 38(B_3+1)\right) = 2 > 0$$

Thus, for $n \in \{0, 2, 3, 4, \dots\}$ we get

$$\frac{1}{B_n+1} - \frac{1}{B_{n+2}} - \frac{1}{B_{n+3}} - \frac{1}{B_{n+2}+1} < 0 \tag{3}$$

By applying inequality (3) repeatedly for $n > 3$, we have

$$\begin{aligned} \frac{1}{B_{n-2}+1} &< \frac{1}{B_n} + \frac{1}{B_{n+1}} + \frac{1}{B_n+1} \\ &< \frac{1}{B_n} + \frac{1}{B_{n+1}} + \left(\frac{1}{B_{n+2}} + \frac{1}{B_{n+3}} + \right. \\ &\quad \left. + \frac{1}{B_{n+2}+1}\right) \\ &< \dots \\ &< \frac{1}{B_n} + \frac{1}{B_{n+1}} + \frac{1}{B_{n+2}} + \frac{1}{B_{n+3}} + \dots \end{aligned}$$

Thus, we obtain

$$\sum_{k=n}^{\infty} \frac{1}{B_k} > \frac{1}{B_{n-2}+1}$$

(b) In a similar way, we will get

$$\begin{aligned} &\frac{1}{B_n-1} - \frac{1}{B_{n+2}} - \frac{1}{B_{n+3}} - \frac{1}{B_{n+2}-1} \\ &= \frac{B_{n+2}^2 + B_{n+3}^2 - B_{n+4}}{B_{n+2}B_{n+3}(B_n-1)(B_{n+2}-1)} + \\ &\quad + \frac{2(-1)^{n+1}19(B_{n+2}-1)}{B_{n+2}B_{n+3}(B_n-1)(B_{n+2}-1)} \end{aligned} \tag{4}$$

If n is odd, then the right-hand side of identity (4) will be positive.

If n is even, then the right-hand side of identity (4) will also be positive, except for $n = 0$ because

$$B_2^2 + B_3^2 - B_4 + 38(B_2+1) = -96 < 0$$

Thus, for $n \geq 1$ we get

$$\frac{1}{B_n-1} - \frac{1}{B_{n+2}} - \frac{1}{B_{n+3}} - \frac{1}{B_{n+2}-1} > 0 \tag{5}$$

By applying inequality (5) repeatedly for $n \geq 3$, we have

$$\begin{aligned} \frac{1}{B_{n-2}-1} &> \frac{1}{B_n} + \frac{1}{B_{n+1}} + \frac{1}{B_n-1} \\ &> \frac{1}{B_n} + \frac{1}{B_{n+1}} + \left(\frac{1}{B_{n+2}} + \frac{1}{B_{n+3}} + \right. \\ &\quad \left. + \frac{1}{B_{n+2}-1}\right) \\ &> \dots \\ &> \frac{1}{B_n} + \frac{1}{B_{n+1}} + \frac{1}{B_{n+2}} + \frac{1}{B_{n+3}} + \dots \end{aligned}$$

Thus, we obtain

$$\sum_{k=n}^{\infty} \frac{1}{B_k} < \frac{1}{B_{n-2}-1}$$

The proof is complete. ■

The proof for Theorem 1 is as following.

Proof: By Lemma 2.(a) and 3.(a) if n is odd and $n \geq 3$, then

$$\frac{1}{B_{n-2}+1} < \sum_{k=n}^{\infty} \frac{1}{B_k} < \frac{1}{B_{n-2}}$$
$$B_{n-2} < \left(\sum_{k=n}^{\infty} \frac{1}{B_k}\right)^{-1} < B_{n-2}+1$$
$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k}\right)^{-1} \right\rfloor = B_{n-2}$$

By Lemma 2.(b) and 3.(b) if n is even and $n \geq 4$, then

$$\frac{1}{B_{n-2}} < \sum_{k=n}^{\infty} \frac{1}{B_k} < \frac{1}{B_{n-2}-1}$$
$$B_{n-2}-1 < \left(\sum_{k=n}^{\infty} \frac{1}{B_k}\right)^{-1} < B_{n-2}$$
$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k}\right)^{-1} \right\rfloor = B_{n-2}-1$$

The proof is complete. ■

The next result that we obtain is for the reciprocal sums of square of generalized Fibonacci-Like sequence.

Theorem 2: If $s = 2$, then

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k^2}\right)^{-1} \right\rfloor = \begin{cases} B_{n-1}B_n - 7, & \text{n is odd and } n \geq 1 \\ B_{n-1}B_n + 6, & \text{n is even and } n \geq 2 \end{cases}$$

Proof: First, we observe the following identity.

$$\begin{aligned}
&\frac{1}{B_{n-1}B_n - 6} - \frac{1}{B_n^2} - \frac{1}{B_nB_{n+1} - 6} \\
&= \frac{B_nB_{n+1} - B_{n-1}B_n}{(B_{n-1}B_n - 6)(B_nB_{n+1} - 6)} - \frac{1}{B_n^2} \\
&= \frac{B_n^2}{(B_{n-1}B_n - 6)(B_nB_{n+1} - 6)} - \frac{1}{B_n^2} \\
&= \frac{B_n^4 - (B_{n-1}B_n - 6)(B_nB_{n+1} - 6)}{B_n^2(B_{n-1}B_n - 6)(B_nB_{n+1} - 6)}
\end{aligned} \tag{6}$$

If n is odd, then the numerator in the right-hand side of identity (6) become

$$\begin{aligned}
&B_n^4 - (B_{n-1}B_n - 6)(B_nB_{n+1} - 6) \\
&= B_n^2(B_n^2 - B_{n-1}B_{n+1}) + 6B_n(B_{n-1} + B_{n+1}) - 36 \\
&= B_n^2\big((-1)^n(5s^2 - 1)\big) + 6B_n(B_{n-1} + B_{n+1}) - 36 \\
&= B_n\big(- (5s^2 - 7)B_n + 6B_{n-1} + 6B_{n-1}\big) - 36 \\
&= B_n\big(- (5s^2 - 7)(B_{n-1} + B_{n-2}) + 12B_{n-1}\big) - 36 \\
&= B_n\big(- (5s^2 - 19)B_{n-1} - (5s^2 - 7)B_{n-2}\big) - 36 \\
&= -B_n\big((5s^2 - 19)B_{n-1} + (5s^2 - 7)B_{n-2}\big) - 36
\end{aligned}$$

If $s \geq 2$, then $5s^2 - 19 > 5s^2 - 7 > 0$. Therefore

$$\begin{aligned}
&B_n^4 - (B_{n-1}B_n - 6)(B_nB_{n+1} - 6) \\
&= -B_n\big((5s^2 - 19)B_{n-1} + (5s^2 - 7)B_{n-2}\big) - 36 < 0
\end{aligned}$$

Thus, we get

$$\frac{1}{B_{n-1}B_n - 6} - \frac{1}{B_n^2} - \frac{1}{B_nB_{n+1} - 6} < 0 \tag{7}$$

By applying inequality (7) repeatedly for $n \geq 1$, we have

$$\begin{aligned}
\frac{1}{B_{n-1}B_n - 6} &< \frac{1}{B_n^2} + \frac{1}{B_nB_{n+1} - 6} \\
&< \frac{1}{B_n^2} + \left(\frac{1}{B_{n+1}^2} + \frac{1}{B_{n+1}B_{n+2} - 6}\right) \\
&< \dots \\
&< \frac{1}{B_n^2} + \frac{1}{B_{n+1}^2} + \frac{1}{B_{n+2}^2} + \dots
\end{aligned}$$

Thus, we obtain

$$\sum_{k=n}^{\infty} \frac{1}{B_k^2} > \frac{1}{B_{n-1}B_n - 6} \tag{8}$$

Next, we observe the following identity. If $s = 2$, then

$$\begin{aligned}
&\frac{1}{B_{n-1}B_n - 7} - \frac{1}{B_n^2} - \frac{1}{B_{n+1}^2} - \frac{1}{B_{n+1}B_{n+2} - 7} \\
&= \frac{B_n(B_n - B_{n-1}) + 7}{B_n^2(B_{n-1}B_n - 7)} - \frac{B_{n+1}(B_{n+1} + B_{n+2}) - 7}{B_{n+1}^2(B_{n+1}B_{n+2} - 7)} \\
&= \frac{B_nB_{n-2} + 7}{B_n^2(B_{n-1}B_n - 7)} - \frac{B_{n+1}B_{n+3} - 7}{B_{n+1}^2(B_{n+1}B_{n+2} - 7)} \\
&= \frac{B_{n-1}^2 - (-1)^{n-1} \cdot 19 + 7}{B_n^2(B_{n-1}B_n - 7)} - \frac{B_{n+2}^2 - (-1)^{n+2} \cdot 19 - 7}{B_{n+1}^2(B_{n+1}B_{n+2} - 7)}
\end{aligned}$$

The numerator on the right-hand side of last identity will become

$$\begin{aligned}
&\big(B_{n-1}^2 - (-1)^{n-1} \cdot 19 + 7\big)\big(B_{n+1}^2(B_{n+1}B_{n+2} - 7)\big) \\
&\quad - \big(B_{n+2}^2 - (-1)^{n+2} \cdot 19 - 7\big)\big(B_n^2(B_{n-1}B_n - 7)\big)
\end{aligned}$$

With some calculation, it can be shown that this numerator will be positive for all $n \geq 1$. Thus, we get

$$\frac{1}{B_{n-1}B_n - 7} - \frac{1}{B_n^2} - \frac{1}{B_{n+1}^2} - \frac{1}{B_{n+1}B_{n+2} - 7} > 0 \tag{9}$$

By applying inequality (9) repeatedly for $n \geq 1$, we get

$$\begin{aligned}
\frac{1}{B_{n-1}B_n - 7} &> \frac{1}{B_n^2} + \frac{1}{B_{n+1}^2} + \frac{1}{B_{n+1}B_{n+2} - 7} \\
&> \frac{1}{B_n^2} + \frac{1}{B_{n+1}^2} + \bigg(\frac{1}{B_{n+2}^2} + \frac{1}{B_{n+3}^2} + \\
&\quad + \frac{1}{B_{n+3}B_{n+4} - 7}\bigg) \\
&> \dots \\
&> \frac{1}{B_n^2} + \frac{1}{B_{n+1}^2} + \frac{1}{B_{n+2}^2} + \frac{1}{B_{n+3}^2} + \dots
\end{aligned}$$

Thus, we obtain

$$\sum_{k=n}^{\infty} \frac{1}{B_k^2} < \frac{1}{B_{n-1}B_n - 7} \tag{10}$$

So, from (8) and (10) we have if $n \geq 1$ and odd, then

$$\frac{1}{B_{n-1}B_n - 6} < \sum_{k=n}^{\infty} \frac{1}{B_k^2} < \frac{1}{B_{n-1}B_n - 7}$$

$$B_{n-1}B_n - 7 < \left(\sum_{k=n}^{\infty} \frac{1}{B_k^2}\right)^{-1} < B_{n-1}B_n - 6$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k^2}\right)^{-1} \right\rfloor = B_{n-1}B_n - 7$$

In a similar way, observe that

$$\begin{aligned}
&\frac{1}{B_{n-1}B_n + 6} - \frac{1}{B_n^2} - \frac{1}{B_nB_{n+1} + 6} \\
&= \frac{B_nB_{n+1} - B_{n-1}B_n}{(B_{n-1}B_n + 6)(B_nB_{n+1} + 6)} - \frac{1}{B_n^2} \\
&= \frac{B_n^2}{(B_{n-1}B_n + 6)(B_nB_{n+1} + 6)} - \frac{1}{B_n^2} \\
&= \frac{B_n^4 - (B_{n-1}B_n + 6)(B_nB_{n+1} + 6)}{B_n^2(B_{n-1}B_n + 6)(B_nB_{n+1} + 6)} \qquad (11)
\end{aligned}$$

If n is even, then the numerator in the right-hand side of identity (11) become

$$\begin{aligned}
&B_n^4 - (B_{n-1}B_n + 6)(B_nB_{n+1} + 6) \\
&= B_n^2(B_n^2 - B_{n-1}B_{n+1}) + 6B_n(B_{n-1} + B_{n+1}) - 36 \\
&= B_n^2\big((-1)^n(5s^2 - 1)\big) + 6B_n(B_{n-1} + B_{n+1}) - 36 \\
&= B_n\big((5s^2 + 5)B_n + 6B_{n-1} + 6B_{n-1}\big) - 36 \\
&= B_n\big((5s^2 + 5)(B_{n-1} + B_{n-2}) + 12B_{n-1}\big) - 36 \\
&= B_n\big((5s^2 + 17)B_{n-1} + (5s^2 + 5)B_{n-2}\big) - 36 \\
&= B_n\big((5s^2 + 17)B_{n-1} + (5s^2 + 5)B_{n-2}\big) - 36 > 0
\end{aligned}$$

Therefore, we have

$$\frac{1}{B_{n-1}B_n + 6} - \frac{1}{B_n^2} - \frac{1}{B_nB_{n+1} + 6} > 0 \qquad (12)$$

By applying inequality (12) repeatedly for $n \geq 2$, we have

$$\begin{aligned}
\frac{1}{B_{n-1}B_n + 6} &> \frac{1}{B_n^2} + \frac{1}{B_nB_{n+1} + 6} \\
&> \frac{1}{B_n^2} + \left(\frac{1}{B_{n+1}^2} + \frac{1}{B_{n+1}B_{n+2} + 6}\right) \\
&> \dots \\
&> \frac{1}{B_n^2} + \frac{1}{B_{n+1}^2} + \frac{1}{B_{n+2}^2} + \dots
\end{aligned}$$

Thus, we obtain

$$\sum_{k=n}^{\infty} \frac{1}{B_k^2} < \frac{1}{B_{n-1}B_n + 6} \qquad (13)$$

Next, we have if $s = 2$, then

$$\begin{aligned}
&\frac{1}{B_{n-1}B_n + 7} - \frac{1}{B_n^2} - \frac{1}{B_{n+1}^2} - \frac{1}{B_{n+1}B_{n+2} + 7} \\
&= \frac{B_n(B_n - B_{n-1}) - 7}{B_n^2(B_{n-1}B_n + 7)} - \frac{B_{n+1}(B_{n+1} + B_{n+2}) + 7}{B_{n+1}^2(B_{n+1}B_{n+2} + 7)} \\
&= \frac{B_nB_{n-2} - 7}{B_n^2(B_{n-1}B_n + 7)} - \frac{B_{n+1}B_{n+3} + 7}{B_{n+1}^2(B_{n+1}B_{n+2} + 7)} \\
&= \frac{B_{n-1}^2 - (-1)^{n-1} \cdot 19 - 7}{B_n^2(B_{n-1}B_n + 7)} - \frac{B_{n+2}^2 - (-1)^{n+2} \cdot 19 + 7}{B_{n+1}^2(B_{n+1}B_{n+2} + 7)}
\end{aligned}$$

The numerator on the right-hand side of last identity will become

$$\begin{aligned}
&\big(B_{n-1}^2 - (-1)^{n-1} \cdot 19 - 7\big)\big(B_{n+1}^2(B_{n+1}B_{n+2} - 7)\big) \\
&\quad -\big(B_{n+2}^2 - (-1)^{n+2} \cdot 19 - 7\big)\big(B_n^2(B_{n-1}B_n + 7)\big)
\end{aligned}$$

With some calculation, it can be shown that this numerator will be negative for all $n \geq 1$. Thus, we get

$$\frac{1}{B_{n-1}B_n + 7} - \frac{1}{B_n^2} - \frac{1}{B_{n+1}^2} - \frac{1}{B_{n+1}B_{n+2} + 7} < 0 \qquad (14)$$

By applying inequality (14) repeatedly for $n \geq 1$, we get

$$\begin{aligned}
\frac{1}{B_{n-1}B_n + 7} &< \frac{1}{B_n^2} + \frac{1}{B_{n+1}^2} + \frac{1}{B_{n+1}B_{n+2} + 7} \\
&< \frac{1}{B_n^2} + \frac{1}{B_{n+1}^2} + \bigg(\frac{1}{B_{n+2}^2} + \frac{1}{B_{n+3}^2} + \\
&\quad + \frac{1}{B_{n+3}B_{n+4} + 7}\bigg) \\
&< \dots \\
&< \frac{1}{B_n^2} + \frac{1}{B_{n+1}^2} + \frac{1}{B_{n+2}^2} + \frac{1}{B_{n+3}^2} + \dots
\end{aligned}$$

Thus, we obtain

$$\sum_{k=n}^{\infty} \frac{1}{B_k^2} > \frac{1}{B_{n-1}B_n + 7} \qquad (15)$$

So, from (13) and (15) we have if $n \geq 1$ and even, then

$$\frac{1}{B_{n-1}B_n + 7} < \sum_{k=n}^{\infty} \frac{1}{B_k^2} < \frac{1}{B_{n-1}B_n + 6}$$

$$B_{n-1}B_n + 6 < \left(\sum_{k=n}^{\infty} \frac{1}{B_k^2}\right)^{-1} < B_{n-1}B_n + 7$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{B_k^2}\right)^{-1} \right\rfloor = B_{n-1}B_n + 6$$

The proof is complete. ■

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