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Global stability in a mathematical model of de-radicalization

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HIGHLIGHTS

- De-radicalization modeled with an epidemic model.
- Five compartments are considered: Susceptible, Recruiters, Extremists and Treatment.
- The dynamics is determined by the basic reproduction number $\mathcal{R}_0$.
- If $\mathcal{R}_0 > 1$ the equilibrium with no terrorists is globally stable, and extremists and recruiters head for extinction.
- Model is used to assess strategies to counter violent extremism.

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ABSTRACT

Radicalization is the process by which people come to adopt increasingly extreme political, social or religious ideologies. When radicalization leads to violence, radical thinking becomes a threat to national security. De-radicalization programs are part of an effort to combat violent extremism and terrorism. This type of initiatives attempt to alter violent extremists radical beliefs and violent behavior with the aim to reintegrate them into society. In this paper we introduce a simple compartmental model suitable to describe de-radicalization programs. The population is divided into four compartments: (S) susceptible, (E) extremists, (R) recruiters, and (T) treatment. We calculate the basic reproduction number $\mathcal{R}_0$. For $\mathcal{R}_0 < 1$ the system has one globally asymptotically stable equilibrium where no extremist or recruiters are present. For $\mathcal{R}_0 > 1$ the system has an additional equilibrium where extremists and recruiters are endemic to the population. A Lyapunov function is used to show that, for $\mathcal{R}_0 > 1$, the endemic equilibrium is globally asymptotically stable. We use numerical simulations to support our analytical results. Based on our model we assess strategies to counter violent extremism.

1. Introduction

According to Horgan [1] radicalization is the social and psychological process of incrementally experienced commitment to extremist political or religious ideology. Radicalization can lead to violent extremism and therefore it has become a major concern for national security. Typical counterterrorism strategies fall into two categories:

1. Law enforcement approach: violent extremist are investigated prosecuted and imprisoned.
2. Military approach: violent extremists are killed or captured on the battlefield.

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Practitioners of counterterrorism agree that these approaches alone cannot break the cycle of violence [2]. The realization of the inadequacy of the counterterrorism approach has lead to different strategies, collectively known as countering violent extremism (CVE). CVE is a collection of noncoercive activities whose aim is to intervene in an individual's path toward violent extremism, to interdict criminal activity and to reintegrate those convicted of criminal activity into society. CVE programs can be divided into three broad classes [2–5]

1. *Prevention programs*, which seek to prevent the radicalization process from occurring and taking hold in the first place;
2. *Disengagement programs*, which attempt to stop or control radicalization as it is occurring;
3. *De-radicalization programs*, which attempt to alter an individual extremist beliefs and violent behavior with the aim to reintegrate him into society. This type of programs often target convicted terrorists.

According to Horgan [6] there are at least 15 publicly known de-radicalization programs from Saudi Arabia to Singapore, but there are likely twice as many. In this paper we use a compartmental model to model de-radicalization programs.

The attempt to use quantitative methods in describing social dynamics is not new, and compartmental models have been used to study various aspect of social dynamics. For instance Hayward introduced a model of church growth [7], Jeffs et al. studied a model of political party growth [8], Romero et al. analyzed a model for the spread of political third parties [9] and Crisosto et al. studied the growth of cooperative learning in large communities [10]. The dynamics of the spread of crime was studied by McMillon, Simon and Morenoff [11] and by Mohammad and Roslan [12]. A mathematical model of the spread of gangs was studied by Sooknanan, Bhatt, and Comissiong [13]. The same authors studied the model for the interaction of police and gangs in [14]. Castillo-Chavez and Song analyzed the transmission dynamics of fanatic behaviors [15], Camacho studied a model of the interaction between terrorist and fanatic groups [16], Nizamani, Memon and Galam modeled public outrage and the spread of violence [17]. Compartmental models of radicalization were studied by Galam and Javarone [18] and by McCluskey and Santoprete [19].

In this paper we build on the compartmental model introduced in [19] by adding a treatment compartment. This allows us to consider de-radicalization in our analysis. We divide the population into four compartments, (S) susceptible, (E) extremists, (R) recruiters, and (T) treatment (see Fig. 1). Using this simple model, we attempt to test the effectiveness of de-radicalization programs in countering violent extremism. This is an important issue since, at least on the surface, these de-radicalization programs are promising. In fact, these programs appear to be cost effective, since they are far cheaper than indefinite detention [6]. However, the degree of government support for these programs hinges on their efficacy and, unfortunately, indicators of success and measures of effectiveness remain elusive [3].

As in [19] we use the basic reproduction number $\mathcal{R}_0$ to evaluate strategies for countering violent extremism. We will show that for $\mathcal{R}_0 < 1$ the system has a globally asymptotically stable equilibrium with no individuals in the extremist, recruiter and treatment classes, and that for $\mathcal{R}_0 > 1$ the system has an additional equilibrium in which extremists and recruiters are endemic to the population. The latter equilibrium is globally asymptotically stable for $\mathcal{R}_0 > 1$. Therefore, if $\mathcal{R}_0 < 1$ the ideology will be eradicated, that is, eventually the number of recruiters and extremists will go to zero. When $\mathcal{R}_0 > 1$ the ideology will become endemic, that is, the recruiters and extremists will establish themselves in the population. In our model the basic reproduction number is

$$\mathcal{R}_0 = \frac{\Lambda}{\mu} \frac{\beta\left(c_E q_E + b_E q_R - \frac{(1-k)\delta p_E}{b_T} q_R\right)}{b_E b_R - c_E c_R - \frac{(1-k)\delta}{b_T}(c_E p_R + b_R p_E)}, \tag{1.1}$$

where μ is the mortality rate of the susceptible population, k is the fraction of successfully de-radicalized individuals, and δ is the rate at which individuals leave the treatment compartment, so that $1/\delta$ is the average time spent in the treatment compartment. The rates at which extremists and recruiters enter the treatment compartment are p_E and p_R, respectively. Moreover, $b_E = \mu + d_E + c_E + p_E$ and $b_R = \mu + d_R + c_R + p_R$, where d_E and d_R are the additional mortality rates of the extremists and recruiters, respectively.[1] Other parameters are described in Section 2. Note that, if $p_E, p_R \to 0$, then the basic reproduction number limits to the one of the bare-bones model studied in [19].

One approach to dealing with extremism, which follows under the umbrella of counterterrorism, is to prosecute and imprison violent extremists. This approach was studied in [19] where it was shown that increasing the parameters d_E and d_R resulted in a decrease in $\mathcal{R}_0$. A similar results holds for the model studied in this paper. A different strategy consists in improving the de-radicalization programs by either increasing the success rate k or by increasing the rates p_E and p_R at which extremists and recruiters enter the T compartment. Since $\mathcal{R}_0$ is a decreasing function of k, p_E, and p_R, increasing these parameters decreases $\mathcal{R}_0$. Hence, according to our model, this is a successful strategy to counter violent extremism. Another option is to decrease δ, which in turn decreases $\mathcal{R}_0$. This approach is also viable because $\mathcal{R}_0$ is an increasing function of δ. A good way of thinking about this is to consider prison-based de-radicalization programs, in which case, decreasing δ corresponds to increasing $\frac{1}{\delta}$, the average prison sentence.

Note that, in general, it may not be easy to determine the values of parameters because available data are scarce. It has been claimed, however, that the de-radicalization program in Saudi Arabia, has a rate of recidivism of about 10–20% [6], which gives an estimate for the value of k.

[1] In the context of the present model these can be viewed as the rates at which extremists and recruiters are imprisoned with life sentences.

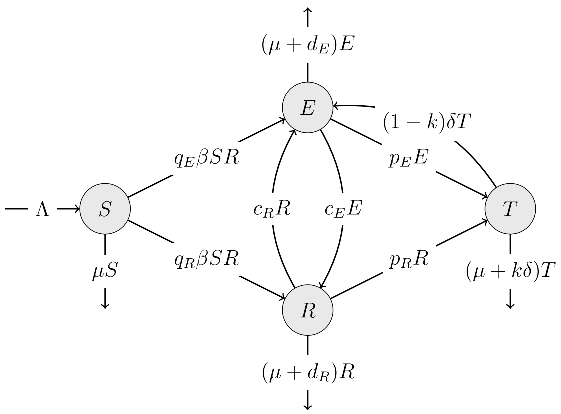

Fig. 1. Transfer diagram for the de-radicalization model.

The paper is organized as follows. In Section 2 we introduce the mathematical model. In Section 3 we find an equilibrium with no individuals in the extremists, recruiters and treatment compartments. We also compute the basic reproduction number using the next generation method. In Section 4 we use Lyapunov functions to prove this critical point is globally asymptotically stable for $\mathcal{R}_0 < 1$. In Section 5 we find another equilibrium point, the *endemic equilibrium*, and we prove it is globally asymptotically stable for $\mathcal{R}_0 > 1$. In Section 6 we present some numerical simulations supporting our analytical results. The final section concludes the paper with a short summary and discussion of the results, limitations of our model and ideas for future research.

2. Equations

We model the spread of extreme ideology as a contact process. We assume that within the full population there is a subpopulation potentially at risk of adopting the ideology. We partition this subpopulation into four compartments:

1. (S) Susceptible
2. (E) Extremists
3. (R) Recruiters
4. (T) Treatment.

Our model is based on the bare-bones mathematical model of radicalization introduced in [19]. Here, however, we also include a treatment compartment (T), to describe de-radicalized individuals. The transfer diagram for this system is given below.

We assume that susceptibles and recruiters interact according to a mass action law, and that the rate at which susceptibles are recruited to adopt the extremist ideology is proportional to the number of interactions that are occurring. Thus, susceptibles are recruited at rate βSR, with a fraction q_E entering the extremist class and a fraction $q_R = 1 - q_E$ entering the recruiter class. Extremists switch to the recruiter class with rate constant c_E, while recruiters enter the extremist class with rate constant c_R. The natural death rate is proportional to the population size, with rate constant μ. Extremists and recruiters have additional death rates d_E and d_R, respectively. These rates account for individuals that are imprisoned for life or killed. To consider individuals that undergo de-radicalization program, extremists and recruiters are made to enter the treatment compartment at rate constants p_E and p_R respectively. The rate at which a treated individual leaves the compartment T is δ. A fraction $k \in [0, 1]$ of treated individuals is removed, since we assume that successfully treated individuals are permanently de-radicalized. This seems to be a reasonable assumption since, according to Horgan [6], individuals who leave terrorism behind have a low chance of re-engagement. The fraction of individuals for which the de-radicalization program fails is $1-k$. These individuals enter the extremist class E after being treated. Thus, the radicalization model consists of the following differential equations together with non-negative initial conditions:

$$\begin{aligned} S' &= \Lambda - \mu S - \beta SR \\ E' &= q_E\beta SR - (\mu + d_E + c_E + p_E)E + c_R R + (1-k)\,\delta T \\ R' &= q_R\beta SR + c_E E - (\mu + d_R + c_R + p_R)R \\ T' &= p_E E + p_R R - (\mu + \delta)T \end{aligned} \tag{2.1}$$

where $q_E + q_R = 1, q_E, q_R \in [0, 1]$. For simplicity denote $b_E = \mu + d_E + c_E + p_E$, $b_R = \mu + d_R + c_R + p_R$ and $b_T = \mu + \delta$, then system (2.1) takes the following form:

$$\begin{aligned}
S' &= \Lambda - \mu S - \beta SR \\
E' &= q_E \beta SR - b_E E + c_R R + (1-k)\,\delta T \\
R' &= q_R \beta SR + c_E E - b_R R \\
T' &= p_E E + p_R R - b_T T
\end{aligned} \tag{2.2}$$

Proposition 2.1. *The region $\Delta = \left\{(S, E, R, T) \in \mathbb{R}^4_{\geq 0} : S + E + R + T \leq \frac{\Lambda}{\mu}\right\}$ is a compact positively invariant set for the flow of (2.1) (i.e. all solutions starting in Δ remain in Δ for all $t > 0$). Moreover, Δ is attracting within $\mathbb{R}^4_{\geq 0}$ (i.e. solutions starting outside Δ either enter or approach Δ in the limit).*

Proof. It is trivial to check that Δ is compact. We first show that $\mathbb{R}^4_{\geq 0}$ is positively invariant by checking the direction of the vector field along the boundary of $\mathbb{R}^4_{\geq 0}$. Along $S = 0$ we have $S' = \Lambda > 0$ so the vector field points inwards. Along $E = 0$ we have $E' = q_E \beta SR + c_R R + (1-k)\delta T \geq 0$, provided $R, S, T \geq 0$. Moreover, along $R = 0$, we have that $R' = c_E E \geq 0$ provided $E \geq 0$. Moreover, along $T = 0$ we have $p_E E + p_R R \geq 0$, provided $E, R \geq 0$. This shows that $\mathbb{R}^4_{\geq 0}$ is positively invariant by Proposition 2.1 in [20]. Now let $N = S + E + R + T$, then

$$S' + E' + R' + T' = \Lambda - \mu N - d_E E - d_R R - k\delta T \leq \Lambda - \mu N.$$

Using a standard comparison theorem, it follows that

$$N(t) \leq \left(N(0) - \frac{\Lambda}{\mu}\right) e^{-\mu t} + \frac{\Lambda}{\mu}, \tag{2.3}$$

for $t \geq 0$. Thus, if $N(0) \leq \frac{\Lambda}{\mu}$, then $N(t) \leq \frac{\Lambda}{\mu}$ for all $t \geq 0$. Hence, the set Δ is positively invariant. Furthermore, it follows from (2.3) that $\limsup_{t\to\infty} N \leq \frac{\Lambda}{\mu}$, demonstrating that Δ is attracting within $\mathbb{R}^4_{\geq 0}$. □

3. Radicalization-free equilibrium and basic reproduction number $\mathcal{R}_0$

If $E = R = T = 0$, then an equilibrium is given by $x_0 = (S_0, E_0, R_0, T_0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$.

The basic reproduction number $\mathcal{R}_0$ is the spectral radius of the next generation matrix G calculated at x_0. $\mathcal{R}_0$ can be calculated as follows (see [21] for more details). In our case the infected compartments are E, R, T. The next generation matrix is given by $G = FV^{-1}$ with

$$F = \begin{bmatrix} \frac{\partial \mathcal{F}_E}{\partial E} & \frac{\partial \mathcal{F}_E}{\partial R} & \frac{\partial \mathcal{F}_E}{\partial T} \\ \frac{\partial \mathcal{F}_R}{\partial E} & \frac{\partial \mathcal{F}_R}{\partial R} & \frac{\partial \mathcal{F}_R}{\partial T} \\ \frac{\partial \mathcal{F}_T}{\partial E} & \frac{\partial \mathcal{F}_T}{\partial R} & \frac{\partial \mathcal{F}_T}{\partial T} \end{bmatrix}(x_0) \quad \text{and} \quad V = \begin{bmatrix} \frac{\partial \mathcal{V}_E}{\partial E} & \frac{\partial \mathcal{V}_E}{\partial R} & \frac{\partial \mathcal{V}_E}{\partial T} \\ \frac{\partial \mathcal{V}_R}{\partial E} & \frac{\partial \mathcal{V}_R}{\partial R} & \frac{\partial \mathcal{V}_R}{\partial T} \\ \frac{\partial \mathcal{V}_T}{\partial E} & \frac{\partial \mathcal{V}_T}{\partial R} & \frac{\partial \mathcal{V}_T}{\partial T} \end{bmatrix}(x_0).$$

Here, $\mathcal{F}_E, \mathcal{F}_R$ and $\mathcal{F}_T$ are the rates of appearance of newly radicalized individuals in the classes E, R, and T, respectively. Let $\mathcal{V}_j = \mathcal{V}_j^- - \mathcal{V}_j^+$, with $\mathcal{V}_j^+$ is the rate of transfers of individuals into class j by all other means, and $\mathcal{V}_j^-$ is the rate of transfers of individuals out of class j, where $j \in \{E, R, T\}$. In our case

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}_E \\ \mathcal{F}_R \\ \mathcal{F}_T \end{bmatrix} = \beta S \begin{bmatrix} q_E R \\ q_R R \\ 0 \end{bmatrix}$$

and

$$\mathcal{V} = \begin{bmatrix} \mathcal{V}_E \\ \mathcal{V}_R \\ \mathcal{V}_T \end{bmatrix} = \begin{bmatrix} b_E E - c_R R - (1-k)\delta T \\ b_R R - c_E E \\ b_T T - (p_E E + p_R R) \end{bmatrix}.$$

Hence

$$F = \beta S_0 \begin{bmatrix} 0 & q_E & 0 \\ 0 & q_R & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} b_E & -c_R & -\alpha_E \\ -c_E & b_R & 0 \\ -p_E & -p_R & b_T \end{bmatrix}.$$

Therefore, the next generation matrix is

$$G = -\frac{S_0\beta}{b_T D}\begin{bmatrix}0 & q_E & 0\\ 0 & q_R & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-b_R b_T & -(\alpha_E p_R + c_R b_T) & \alpha_E b_R\\ -c_E b_T & \alpha_E p_E - b_E b_T & -\alpha_E c_E\\ -b_R p_E - c_E p_R & -b_E p_R - c_R p_E & -b_E b_R + c_R c_E\end{bmatrix}$$
$$= -\frac{\beta S_0}{b_T D}\begin{bmatrix}-q_E c_E b_T & q_E(\alpha_E p_E - b_E b_T) & -q_E \alpha_E c_E\\ -q_R c_E b_T & q_R(\alpha_E p_E - b_E b_T) & -q_R\alpha_E c_E\\ 0 & 0 & 0\end{bmatrix},$$

where $D = b_E b_R - c_E c_R - \frac{\alpha_E}{b_T}(p_E b_R + c_E p_R)$ and $\alpha_E = (1-k)\delta$. Note that F has rank 1 and so the same is true for G. Since two eigenvalues of G are zero the spectral radius is equal to the absolute value of the remaining eigenvalue. Since the trace is equal to the sum of the eigenvalues and there is only one non-zero eigenvalue, we see that the spectral radius of G is equal to the absolute value of the trace (which happens to be positive). Thus,

$$\mathcal{R}_0 = \frac{\beta S_0(c_E q_E + b_E q_R - \frac{\alpha_E p_E}{b_T} q_R)}{b_E b_R - c_E c_R - \frac{\alpha_E}{b_T}(c_E p_R + b_R p_E)}. \tag{3.1}$$

4. Global asymptotic stability of x_0 for $\mathcal{R}_0 < 1$

In this section, we investigate the stability of the critical point x_0. The next generation method provides us with information on the local stability: x_0 is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$. The global asymptotical stability of x_0, instead, is given by the following theorem.

Theorem 4.1. *If $\mathcal{R}_0 \leq 1$ then x_0 is globally asymptotically stable on $\mathbb{R}^4_{\geq 0}$.*

Proof. Consider the C^1 Lyapunov function $U : \Delta \to \mathbb{R}$

$$U = b_T c_E E + (b_T b_E - \alpha_E p_E)R + \alpha_E c_E T,$$

where $(b_T b_E - \alpha_E p_E) = (\mu+\delta)(\mu + d_E + c_E) + \mu p_E + k\delta p_E > 0$. Evaluating the time derivative of U along the trajectories of (2.2) yields

$$\begin{aligned}
U' =& b_T c_E E' + (b_T b_E - \alpha_E p_E)R' + \alpha_E c_E T'\\
=& b_T c_E(q_E \beta S R - b_E E + c_R R + \alpha_E T) + (b_T b_E - \alpha_E p_E)(q_R\beta S R + c_E E - b_R R)\\
&+ \alpha_E c_E(p_E E + p_R R - b_T T)\\
=& b_T\left[\beta(q_E c_E + q_R b_E - q_R\frac{\alpha_E}{b_T}p_E)S - \left(b_E b_R - c_E c_R - \frac{\alpha_E}{b_T}(p_E b_R + c_E p_R)\right)\right]R\\
=& b_T D\left[\beta(q_E c_E + q_R b_E - q_R\frac{\alpha_E}{b_T}p_E)\frac{S}{D} - 1\right]R\\
=& b_T D\left[\mathcal{R}_0\frac{S}{S_0} - 1\right]R.
\end{aligned}$$

It follows from $S \leq S_0 = \frac{\Lambda}{\mu}$ that

$$U' \leq b_T D[\mathcal{R}_0 - 1]R$$

which implies that $U' \leq 0$ if $\mathcal{R}_0 \leq 1$. Furthermore, $U' = 0$ if and only if $\mathcal{R}_0 = 1$ or $R = 0$. Let

$$Z = \{(S, E, R, T) \in \Delta \,|\, U' = 0\}.$$

We claim that the largest invariant set contained in Z is x_0. In fact, any entire solution $(S(t), E(t), R(t), T(t))$ contained in Z must have $R(t) \equiv 0$ as a consequence of the expression for U' given above. Moreover, from the second and third line in (2.2) it follows that $E(t) \equiv 0$ and $T(t) \equiv 0$. Substituting $R = T = 0$ in the first line of (2.2) gives a differential equation with solution $S = \left(S(0) - \frac{\Lambda}{\mu}\right)e^{-\mu t} + \frac{\Lambda}{\mu}$. Clearly, if $S(0) \leq \frac{\Lambda}{\mu}$, then $S \to -\infty$ as $t \to -\infty$ and the corresponding entire solution is not contained in Z. It follows that $S(0) = \frac{\Lambda}{\mu}$, which proves the claim.

Since Δ is positively invariant with respect to (2.2) LaSalle's invariance principle ([22] Theorem 4.4 or [23] Theorem 6.4) implies that all trajectories that start in Δ approach x_0 when $t \to \infty$. This, together with the fact that x_0 is Lyapunov stable (in fact is locally asymptotically stable by the next generation method), proves that x_0 is globally asymptotically stable in Δ. Since Δ is an attracting set within $\mathbb{R}^4_{\geq 0}$ the stability is also global in $\mathbb{R}^4_{\geq 0}$. □

5. Global asymptotic stability of the endemic equilibrium

In this section, we show that if $\mathcal{R}_0 > 1$, then (2.2) has a unique endemic equilibrium. We then study the global asymptotic stability of such equilibrium using Lyapunov functions.

An endemic equilibrium $x^* = (S^*, E^*, R^*, T^*) \in \mathbb{R}^4_{>0}$ of (2.2) is an equilibrium in which at least one of E^*, R^* and T^* is nonzero.

To find the endemic equilibria of (2.2) we need to solve the following system of equations:

$$\begin{aligned}
0 &= \Lambda - \mu S^* - \beta S^* R^* \\
0 &= q_E \beta S^* R^* - b_E E^* + c_R R^* + (1-k)\,\delta T^* \\
0 &= q_R \beta S^* R^* + c_E E^* - b_R R^* \\
0 &= p_E E^* + p_R R^* - b_T T^*
\end{aligned} \tag{5.1}$$

From the last equation we obtain $T^* = \frac{p_E}{b_T} E^* + \frac{p_R}{b_T} R^*$. Using the expression above for T^*, the first two lines of (5.1) and treating S^* as a parameter yields the linear system

$$\begin{bmatrix} -b_E + \dfrac{p_E}{b_T}(1-k)\delta & q_E \beta S^* + c_R + \dfrac{p_R}{b_T}(1-k)\delta \\ c_E & q_R \beta S^* - b_R \end{bmatrix} \begin{bmatrix} E^* \\ R^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.2}$$

In order to have non-zero solutions for E^* and R^*, the coefficient matrix must have determinant zero. This gives

$$S^* = \frac{b_E b_R - c_E c_R - \frac{\alpha_E}{b_T}(c_E p_R + b_R p_E)}{\beta(c_E q_E + b_E q_R - \frac{\alpha_E p_E}{b_T} q_R)} = \frac{\Lambda}{\mu} \frac{1}{\mathcal{R}_0}, \tag{5.3}$$

where $\alpha_E = (1-k)\delta$. Solving the third line of Eq. (5.1) for E^* yields

$$E^* = \omega R^*, \quad \text{with} \quad \omega = \frac{b_R - q_R \beta S^*}{c_E}.$$

Note that

$$\omega = \frac{b_R b_T q_E + b_T c_R q_R + p_R q_R \alpha_E}{q_R(b_E b_T - \alpha_E p_E) + c_E q_E b_T} = \frac{b_R b_T q_E + b_T c_R q_R + p_R q_R \alpha_E}{q_R((\mu+\delta)(\mu + d_E + c_E) + \mu p_E + k\delta p_E) + c_E q_E b_T} > 0.$$

Next, solving the last line in (5.1) for T^* gives

$$T^* = \frac{p_E \omega + p_R}{b_T} R^*.$$

Substituting this last expression in the first line of (5.1) we obtain

$$R^* = \frac{\Lambda - \mu S^*}{\beta S^*} = \frac{\mu}{\beta}(\mathcal{R}_0 - 1).$$

Since $\omega > 0$, it follows that a meaningful endemic equilibrium with positive S^*, E^*, R^*, and T^* exists if and only if $\mathcal{R}_0 > 1$. When the endemic equilibrium exists, there is only one, denoted by $x^* = (S^*, E^*, R^*, T^*)$, where

$$\begin{aligned}
S^* &= \frac{\Lambda}{\mu}\frac{1}{\mathcal{R}_0}, \\
E^* &= \omega R^* \\
R^* &= \frac{\mu}{\beta}(\mathcal{R}_0 - 1) \\
T^* &= \frac{p_E \omega + p_R}{b_T} R^*.
\end{aligned} \tag{5.4}$$

Theorem 5.1. *If $\mathcal{R}_0 > 1$, then the endemic equilibrium x^* of (2.2) is globally asymptotically stable in $\mathbb{R}^4_{>0}$.*

Proof. We study the global stability of x^* by considering the Lyapunov function

$$V = S^* g\left(\frac{S}{S^*}\right) + a_1 E^* g\left(\frac{E}{E^*}\right) + a_2 R^* g\left(\frac{R}{R^*}\right) + a_3 T^* g\left(\frac{T}{T^*}\right)$$

where $g(x) = x - 1 - \ln x$. Clearly V is C^1, $V(x^*) = 0$, and $V(p) > 0$ for any $p \in \mathbb{R}^4_{>0}$ such that $p \neq x^*$.

Differentiating V along solutions of (2.2) yields

$$V' = \left(1 - \frac{S^*}{S}\right) S' + a_1 \left(1 - \frac{E^*}{E}\right) E' + a_2 \left(1 - \frac{R^*}{R}\right) R' + a_3 \left(1 - \frac{T^*}{T}\right) T'$$

$$
\begin{aligned}
&=\left(1-\frac{S^*}{S}\right)[\Lambda-\mu S-\beta SR]+a_1\left(1-\frac{E^*}{E}\right)[q_E\beta SR-b_EE+c_RR+\alpha_ET]\\
&\quad+a_2\left(1-\frac{R^*}{R}\right)[q_R\beta SR+c_EE-b_RR]+a_3\left(1-\frac{T^*}{T}\right)[p_EE+p_RR-b_TT]\\
&=C-(\mu+a_2\beta q_RR^*)S+(a_1q_E+a_2q_R-1)\beta SR+(-a_1b_E+a_2c_E+a_3p_E)E\\
&\quad+(S^*\beta+a_1c_R-a_2b_R+a_3p_R)R+(a_1\alpha_E-a_3b_T)T-\Lambda\frac{S^*}{S}-a_3p_E\frac{T^*}{T}E-a_2c_E\frac{R^*}{R}E\\
&\quad-a_3p_R\frac{T^*}{T}R-a_1\alpha_EE^*\frac{T}{E}-a_1c_RE^*\frac{R}{E}-a_1\beta q_EE^*\frac{SR}{E}
\end{aligned}
$$

where $C=\Lambda+\mu S^*+a_1b_EE^*+a_2b_RR^*+a_3b_TT^*$. For simplicity, denote $w=\frac{S}{S^*}$, $x=\frac{E}{E^*}$, $y=\frac{R}{R^*}$, and $z=\frac{T}{T^*}$. Then,

$$
\begin{aligned}
V'=&C-(\mu+a_2\beta q_RR^*)S^*w+(a_1q_E+a_2q_R-1)\beta S^*R^*wy+(-a_1b_E+a_2c_E+a_3p_E)E^*x\\
&+(S^*\beta+a_1c_R-a_2b_R+a_3p_R)R^*y+(a_1\alpha_E-a_3b_T)T^*z-\Lambda\frac{1}{w}-a_3p_EE^*\frac{x}{z}\\
&-a_2c_EE^*\frac{x}{y}-a_3p_RR^*\frac{y}{z}-a_1\alpha_ET^*\frac{z}{x}-a_1c_RR^*\frac{y}{x}-a_1\beta q_ES^*R^*\frac{wy}{x}:=G(w,x,y,z).
\end{aligned}
$$

As in [24], we define a set $\mathcal{D}$ of the above terms as follows

$$
\mathcal{D}=\left\{w,x,y,z,wy,\frac{1}{w},\frac{x}{z},\frac{x}{y},\frac{y}{z},\frac{z}{x},\frac{y}{x},\frac{wy}{x}\right\}.
$$

There are at most five subsets associated with $\mathcal{D}$ such that the product of all functions within each subset is equal to one, given by

$$
\left\{w,\frac{1}{w}\right\},\left\{\frac{x}{y},\frac{y}{x}\right\}\left\{\frac{x}{z},\frac{z}{x}\right\},\left\{\frac{z}{x},\frac{y}{z},\frac{x}{y}\right\}\left\{\frac{1}{w},\frac{wy}{x},\frac{x}{y}\right\}.
$$

We associate to these subsets of variables the following terms

$$
\left(2-w-\frac{1}{w}\right),\left(2-\frac{x}{y}-\frac{y}{x}\right),\left(2-\frac{x}{z}-\frac{z}{x}\right),\left(3-\frac{z}{x}-\frac{y}{z}-\frac{x}{y}\right),\left(3-\frac{1}{w}-\frac{x}{y}-\frac{wy}{x}\right).
$$

Following the method used in [24,25] we construct a Lyapunov function as a linear combination of the terms above:

$$
\begin{aligned}
H(w,x,y,z)=&b_1\left(2-w-\frac{1}{w}\right)+b_2\left(2-\frac{x}{y}-\frac{y}{x}\right)+b_3\left(2-\frac{x}{z}-\frac{z}{x}\right)\\
&+b_4\left(3-\frac{z}{x}-\frac{y}{z}-\frac{x}{y}\right)+b_5\left(3-\frac{1}{w}-\frac{x}{y}-\frac{wy}{x}\right),
\end{aligned}
\tag{5.5}
$$

where the coefficients $b_1,\dots,b_5$ are left unspecified. We want to determine suitable parameters $a_i>0$ $(i=1,2,3)$ and $b_k\geq 0$ $(i=1,\dots,5)$ such that $G(w,x,y,z)=H(w,x,y,z)$. Equating the coefficient of like terms in G and H gives the following equations:

$$
\begin{aligned}
w^0:&\quad 2(b_1+b_2+b_3)+3(b_4+b_5)=C\\
w:&\quad b_1=(\mu+a_2\beta q_RR^*)S^*\\
wy:&\quad a_1q_E+a_2q_R-1=0\\
x:&\quad -a_1b_E+a_2c_E+a_3p_E=0\\
y:&\quad S^*\beta+a_1c_R-a_2b_R+a_3p_R=0\\
z:&\quad a_1\alpha_E-a_3b_T=0\\
w^{-1}:&\quad b_1+b_5=\Lambda\\
xz^{-1}:&\quad b_3=a_3p_EE^*\\
xy^{-1}:&\quad b_2+b_4+b_5=a_2c_EE^*\\
yz^{-1}:&\quad b_4=a_3p_RR^*\\
zx^{-1}:&\quad b_3+b_4=a_1\alpha_ET^*\\
yx^{-1}:&\quad b_2=a_1c_RR^*\\
wyx^{-1}:&\quad b_5=\beta a_1q_ES^*R^*.
\end{aligned}
$$

If we take (S^*, E^*, R^*, T^*) at the endemic equilibrium then the linear system above is consistent and has a unique solution with

$$a_1 = \frac{c_E}{c_E q_E + b_E q_R - \frac{p_E}{b_T}\alpha_E q_R}$$
$$a_2 = \frac{1}{q_R} - \frac{\frac{q_E}{q_R} c_E}{c_E q_E + b_E q_R - \frac{p_E}{b_T}\alpha_E q_R}$$
$$a_3 = \frac{\frac{c_E \alpha_E}{b_T}}{c_E q_E + b_E q_R - \frac{p_E}{b_T}\alpha_E q_R},$$

and with $b_1, \dots, b_5 > 0$. By the arithmetic mean-geometric mean inequality each of the terms in (5.5) is less than or equal to zero. Furthermore,

$$\mathcal{M} = \left\{ (S, E, R, T) \in \mathbb{R}^4_{>0} | \ \frac{dV}{dt} = 0 \right\} = \left\{ (S, E, R, T) \in \mathbb{R}^4_{>0} | \ S = S^*, \frac{E}{E^*} = \frac{R}{R^*} = \frac{T}{T^*} \right\}.$$

We claim that the largest invariant set in $\mathcal{M}$ is the set consisting of the endemic equilibrium x^*. In fact, let $(S(t), E(t), R(t), T(t))$ be a complete orbit in $\mathcal{M}$, then

$$0 = S' = (S^*)' = \Lambda - \mu S^* - \beta S^* R,$$

which implies that

$$R = \frac{\Lambda - \mu S^*}{\beta S^*} = R^*.$$

Therefore, $x^* = (S(t), E(t), R(t), T(t))$. By LaSalle's invariance principle [22,23], we deduce that all solutions of (2.2) that start in $\mathbb{R}^4_{>0}$ limit to x^*. This, together with the fact that x^* is Lyapunov stable, prove that x^* is globally asymptotically stable. □

6. Numerical simulations

In this section, we present some numerical simulations of system (2.1) to support our analytical results.

In our simulation, we use the death rate $\mu = 0.000034247\,(\text{days})^{-1}$ [26], i.e., the life expectancy is 80 years. We consider the de-radicalization in a region with population size of about 17.5 million, and thus $\Lambda = 600\,(\text{days})^{-1}$. The other system parameters are chosen, in a somewhat arbitrary manner, to be $\beta = 0.00000000056\,(\text{days})^{-1}$, $d_E = 0.00083\,(\text{days})^{-1}$, $d_R = 0.00083\,(\text{days})^{-1}$, $p_E = 0.00175\,(\text{days})^{-1}$, $p_R = 0.0019\,(\text{days})^{-1}$, $c_E = 0.0006\,(\text{days})^{-1}$, $c_R = 0.0008\,(\text{days})^{-1}$, and $\delta = 0.0016\,(\text{days})^{-1}$. Here, $q_E = 0.86$, $q_R = 0.14$ and $k = 0.66$ are dimensionless, and the time unit is day. In this case we find that $\mathcal{R}_0 = 1.02$, and thus, by Theorem 5.1, the endemic equilibrium x^* is globally asymptotically stable in $\mathbb{R}^4_{>0}$. Figs. 2(a)–(d) depict S, E, R, and T as a function of the time t (days), and show that after a few oscillations these populations approach a constant value. Figs. 2(e) and (f), instead, are phase portraits obtained for different initial conditions. These two figures confirm that the solutions approach a globally asymptotically stable equilibrium point. This case illustrates the unwanted scenario where terrorists and recruiters become endemic to the population.

Second, we increase the rates p_E and p_R at which extremist and recruiters enter the T compartment to $p_E = 0.005\,(\text{days})^{-1}$, and $p_R = 0.006\,(\text{days})^{-1}$ and leave the rest of the parameters unchanged. This can be viewed as an improvement of the recruitment into the de-radicalization programs. Figs. 3(a)–(d) show that $E, R, T \to 0$, as the time t grows large, confirming that x_0 is globally asymptotically stable. This is the preferred situation, where the number of extremists and recruiters decreases to zero.

7. Discussion

In this paper, we presented an abstract compartmental model of radicalization obtained by modifying the one proposed by McCluskey and Santoprete in [19] to include the deradicalization process. Although our abstract model does not generate empirical findings, it does point to some interesting relations between the system parameters and it suggests policies and interventions that reduce the likelihood of radicalization. One advantage of this type of simple models is that it is possible to obtain analytical results that do not rely on the numerical value of parameters, which at the present time is not easy to estimate due to the lack of data.

By means of the next generation method we obtained the basic reproduction number $\mathcal{R}_0$, which plays an important role in controlling the spread of the extremist ideology. By constructing two Lyapunov functions we studied the global stability of the equilibria. We showed that this new model displays a threshold dynamics. When $\mathcal{R}_0 \leq 1$ all solutions converge to the radicalization-free equilibrium, and the populations of recruiters and extremists decreases to zero. When $\mathcal{R}_0 > 1$ the radicalization-free equilibrium is unstable and there is also an additional endemic equilibrium that is globally asymptotically stable. In this case extremists and recruiters will persist in the population.

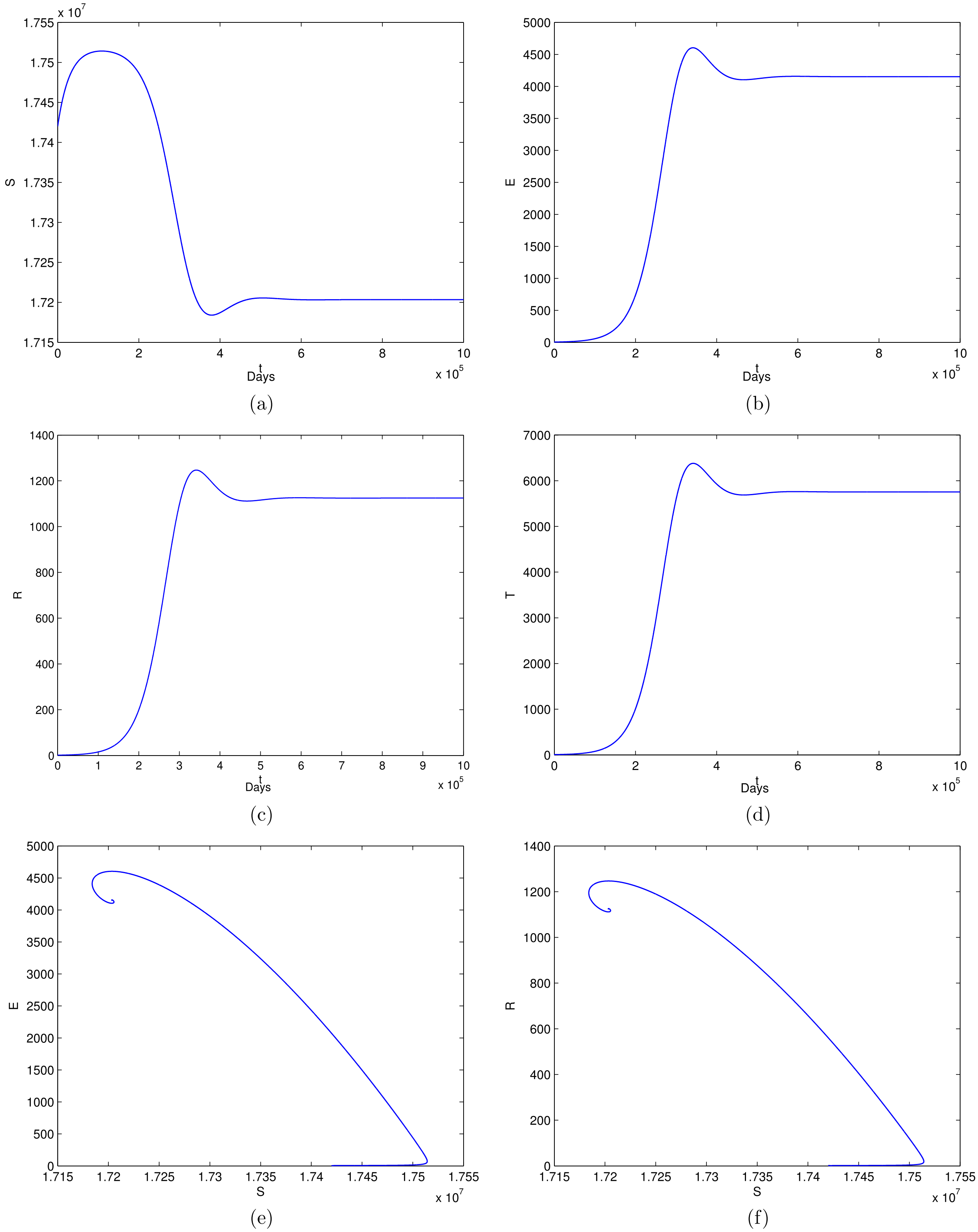

Fig. 2. Time history and phase portraits of system (2.1) for $\beta = 0.00000000056\,(\text{days})^{-1}$, $q_E = 0.86$, $d_E = 0.00083\,(\text{days})^{-1}$, $d_R = 0.00083\,(\text{days})^{-1}$, $p_E = 0.00175\,(\text{days})^{-1}$, $p_R = 0.0019\,(\text{days})^{-1}$, $c_E = 0.0006\,(\text{days})^{-1}$, $c_R = 0.0008\,(\text{days})^{-1}$, $k = 0.66$, $\delta = 0.0016\,(\text{days})^{-1}$, $\mu = 0.000034247\,(\text{days})^{-1}$, $\Lambda = 600\,(\text{days})^{-1}$ and $q_R = 0.14$.

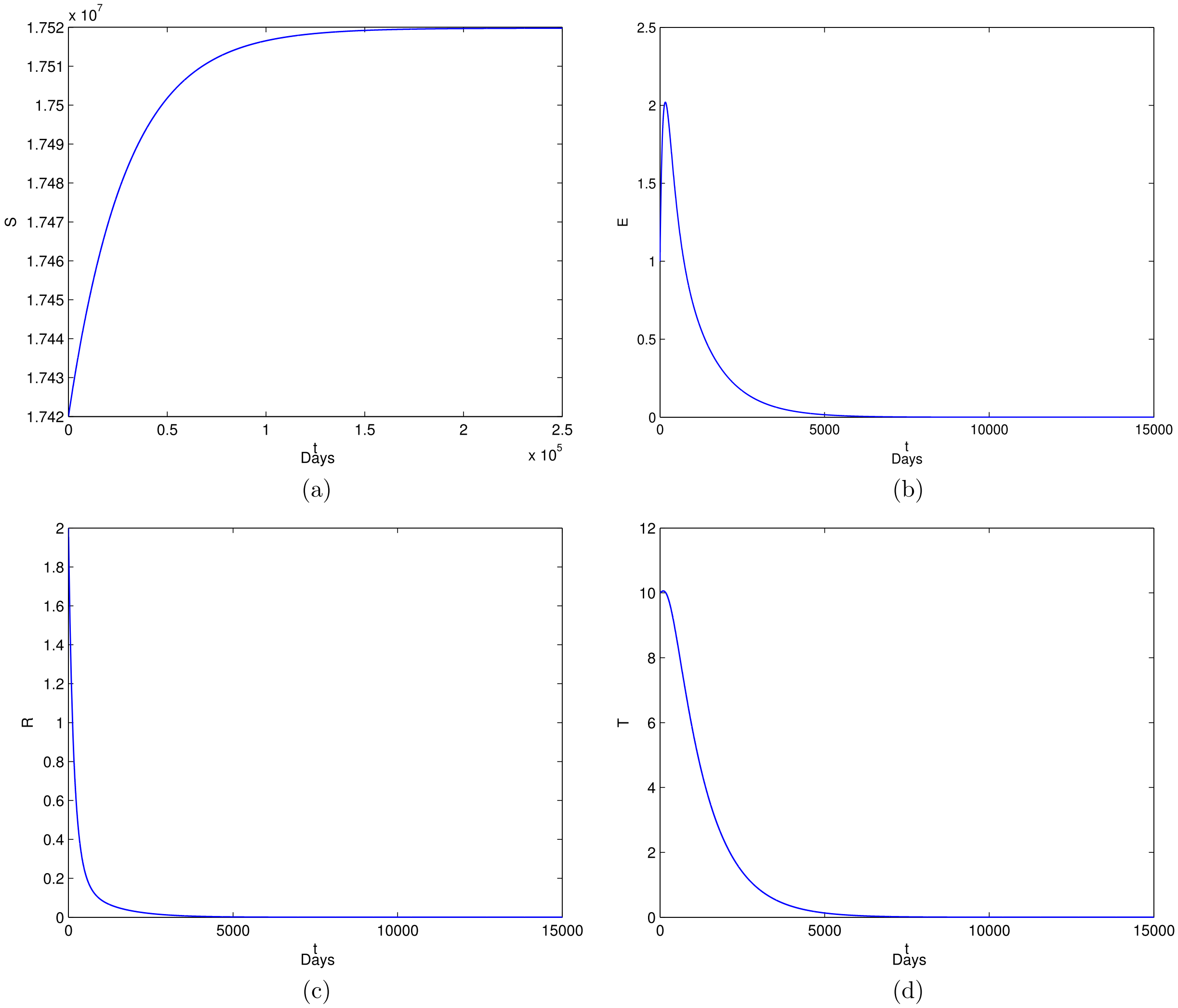

Fig. 3. Time history of system (2.1) for $\beta = 0.00000000056\,(\text{days})^{-1}$, $q_E = 0.86$, $d_E = 0.00083\,(\text{days})^{-1}$, $d_R = 0.00083\,(\text{days})^{-1}$, $p_E = 0.005\,(\text{days})^{-1}$, $p_R = 0.006\,(\text{days})^{-1}$, $c_E = 0.0006\,(\text{days})^{-1}$, $c_R = 0.0008\,(\text{days})^{-1}$, $k = 0.66$, $\delta = 0.0016\,(\text{days})^{-1}$, $\mu = 0.000034247\,(\text{days})^{-1}$, $\Lambda = 600\,(\text{days})^{-1}$ and $q_R = 0.14$.

Based on our model, we can evaluate strategies for countering violent extremism using the basic reproduction number. In particular, our model shows that treatment in the form of de-radicalization programs can be successful in eliminating extremism if used in conjunction with counterterrorism strategies. In fact, it is easy to see that an increase in the success rate k or an increase in the rates p_E and p_R at which extremists and recruiters enter the T compartment causes the basic reproduction number $\mathcal{R}_0$ to decrease. This confirms that, according to our model, de-radicalization programs can be effective. If it is not possible to change k, p_E or p_R it is often possible to decrease δ, which in turn decreases $\mathcal{R}_0$. Note that a decrease in δ corresponds to an increase in $\frac{1}{\delta}$, the average prison sentence, which suggests that increasing the length of the prison sentence can, at least in part, compensate for low success rates or low transfer rates p_E and p_R. Hence, the length of the prison sentence should be chosen so that a perfect balance can be struck between various requirements, including financial ones.

One serious issue of de-radicalization programs is that it is very difficult to evaluate their effectiveness and so estimates of the success rate k are very imprecise. O'Halloran [3] identifies various causes including the lack of empirical evidence and the lack of valid and reliable indicators of de-radicalization. This means that there are issues in evaluating de-radicalization programs that are beyond what can be explored with simple mathematical models. In particular, from the practical point of view, it is unclear how one can improve the success rate k. Furthermore, when modeling social dynamics one has to make many simplifying assumptions. The model studied in this paper is not completely free from this defect. One issue, for instance, is that extremists and recruiters entering the treatment compartment will stay in the compartment for a period of time, given by the length of the prison sentence or of the de-radicalization treatment. Hence, it seems possible to consider more realistic models by using delay differential equations, and include the time of the de-radicalization treatment as a time delay. A further concern is that the population in the various compartments may not be homogeneous. For example, the parameter β may depend on the age of the susceptible, suggesting that an age-structured model may be better suited to describe this problem. We plan to address these and other issues in future studies.

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