RELATIONSHIP BETWEEN STATIC AND DYNAMIC DISPLACEMENTS OF STRUCTURES

by Endah Wahyuni[a]

ABSTRACT

The relationship between the static and dynamic displacements of a structure is studied and explores the application of the relationship. The key parameters are: the static stiffness of the structure, which is a measure of its resistance to an applied load; and the dynamic stiffness, which relates to a specific mode of vibration. To illustrate the salient features, two simple examples will be considered, first a beam and second a plate. These may be considered to be simple representations of a bridge and a floor respectively. The conclusions of the research are the number of modes considered increases the difference between total modal displacement and the static displacement decreases. The first mode dominated the sumof the modal displacement. It is therefore apparent that measurements of both static and dynamic stiffness can both provide useful information on the elastic behavior of a structure.

KEYWORDS: *beam; displacement; dynamic; plates; relationship; static.*

INTRODUCTION

This paper derives the relationship between the static and dynamic displacements of a structure and explores the application of the relationship. Both the displacements indicate the capacity of the structure to resist deformation from different prospects, they are obtained using different methods. The former is calculated by solving the equations of equilibrium and the latter can be obtained through solving an eigen value problem. The concepts need explaining in some detail even to engineers involved with testing structures, as the ideas behind this project are new. To illustrate the salient features, two simple examples will be considered, first a beam and second a plate. These may be considered to be simple representations of a bridge and a floor respectively. Initially numerical models will be examined.

The key parameters are: the static stiffness of the structure, which is a measure of its resistance to an applied load; and the dynamic stiffness, which relates to a specific mode of vibration. The relationship between these parameters will be examined in the next sections, and will also illustrate how some of the factors relate to the structural strength.

FE ANALYSIS OF SIMPLY SUPPORTED BEAMS

There are many factors that are required in the prediction of structural behaviour and the structures static stiffness and dynamic characteristics are important. However, the combined use of static and dynamic characteristics can provide a better understanding of structural behaviour. To illustrate this an analysis of a simple beam is used to compare modal and static stiffnesses for multi-span systems.

When considering dynamic characteristics it is useful to be aware of some basic relationships. The frequency of a mode of vibration of a structure is related to the modal stiffness and mass by the following equation that is reasonably accurate for systems with low damping (ie most structures).

$$2\pi f = \sqrt{\frac{k}{m}} k = m \ 2\pi f^{\ 2} \tag{1}$$

Where f is the natural frequency and k and m are the modal stiffness and modal mass respectively. The modal mass, m is related to the actual mass by the mode shape factor ϕ^2.

The static stiffness can be defined in many ways. In this report point stiffness will be considered, which can be defined as the inverse of the displacement in the load direction at a position where a unit load is applied.

$$k_s = \frac{P}{\Delta} \tag{2}$$

Where k_s is the static stiffness, P is the applied load, Δ is the displacement of the structure.

The point stiffness relates to a unit force that is a function of position and direction; in other words, the point stiffnesses at different positions and directions are different. However, it is usual to apply the point load at the position and direction where it will yield the largest displacement, ie for a plate it would be applied at the centre and normal to the plate.

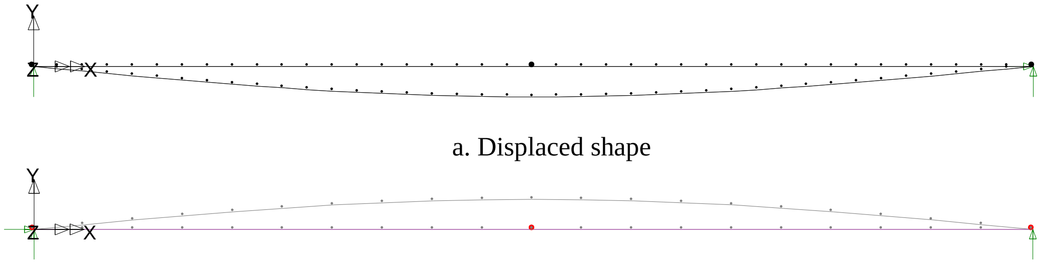
SIMPLY SUPPORTED BEAMS

Consider a simply supported beam, which has a half-sine mode shape for its fundamental mode. Here the modal mass is easy to determine and the system is easy to visualise. The slender beam is 10 m long, of steel construction, and a standard hollow section of

Area	= 0.1420E-3 m^2
I	= 0.7590E-8 m^4
density ρ	= 7800 kg/m^3
E	= 2.09E11 N/m^2
Total mass	= 11.076 kg

[a]Lecturer in the Department of Civil Engineering, Sepuluh Nopember Institute of Technology (ITS), ITS Campus, Sukolilo, Surabaya 60111, Indonesia.

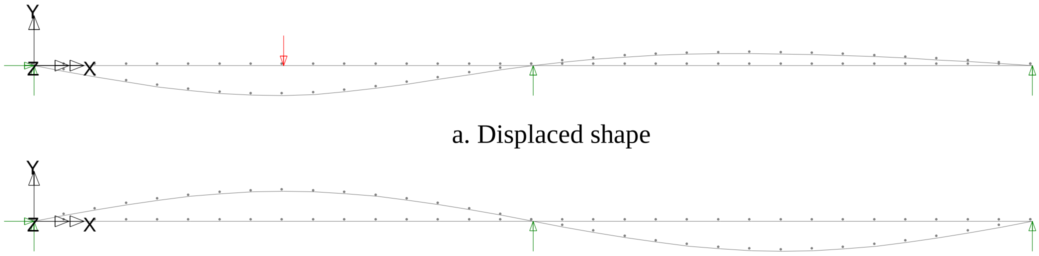
Note. The manuscript for this paper was submitted for review and possible publication on July 06, 2010; approved on August 23, 2010. Discussion open until May 2011. This paper is part of the ITS Journal of Civil Engineering, Vol. 30, No. 2, November 2010. © ITS Journal of Civil Engineering, ISSN 2086-1206/2010.



a. Displaced shape

b. First mode shape

Figure 1. Single span simply supported beam



a. Displaced shape

b. First mode shape

Figure 2. Two-span simply supported beam

Support conditions are pinned at one end and simply supported at the other. From standard texts[1] the formula for the frequencies of the first three modes are:

$$f_1 = \frac{\pi}{2}\sqrt{\frac{EI}{mL^4}} \text{ with } f_2 = 4 f_1 \text{ and } f_3 = 9 f_1 \qquad (3)$$

m (mass per unit length) is Area x ρ = 1.1076 thus f_1 = 0.5945 Hz and the mode shape is a half-sine wave.

f_2 = 2.3786 Hz
f_3 = 5.350 Hz

FE Dynamic Analysis of A Simply Supported Beam

Setting up this type of analysis on a finite element package is relatively easy. For this work the FE package LUSAS was used. The beam length is 10 m, and the supports are defined at positions 0,0 and 10,0. The model uses a beam element with 10 divisions, assigned as a standard SHS steel section, with appropriate boundary conditions. This gives the lowest three frequencies 0.594, 2.378, 5.352 Hz which align with the results from the standard formula as expected.

Given the deflected values of the mode shape for each element node, these can be normalised. For ϕ^2 it is necessary to take the sum of the square of the deflection at each node (including the end nodes) and divided by the number of nodes, with end points having ½ weighting. Then multiply by the total mass and find modal stiffness using the frequency. Using equation (1), this gives a modal stiffness for the fundamental mode of

$$k = m\,(2\pi f)^2 = 11.076 \text{ x}(0.4998\ (2 \text{ xpx } 0.594)^2) \qquad (4)$$
$$= 77.11 \text{ N/m}$$

Obviously knowing the mode shape to be half-sine wave could be used to simplify this procedure significantly. i.e.

$$\int_0^1 \sin^2(\pi x)dx = 0.5 \qquad (5)$$

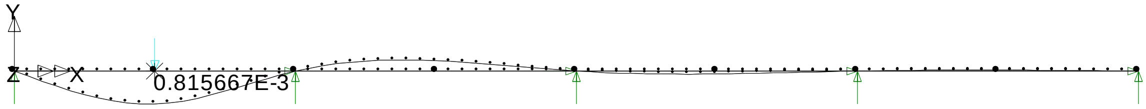
Whereas the numerical approximation gave 0.4998. Numerical evaluation is required for more complex mode shapes that cannot be defined mathematically.

Fe Static Analysis of A Simply Supported Beam

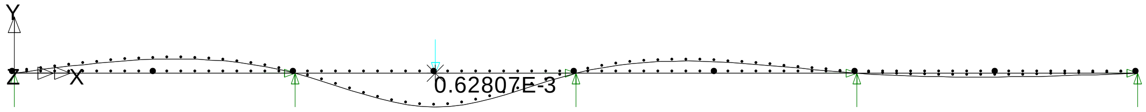
Using the same FE model, apply a unit load at the centre of beam and determine a displaced shape and the maximum displacement. By definition, the inverse of the displacement produced by the unit load is the stiffness, which for the load applied at the centre of the beam gives 76.16 N/m. The displaced shapes for the two static cases and the mode shape are given in Figure 1. It can be appreciated that these deflected shapes are similar. Here the normalised mode shape is divided by the modal stiffness.

FE Analysis of 2 span simply supported beam

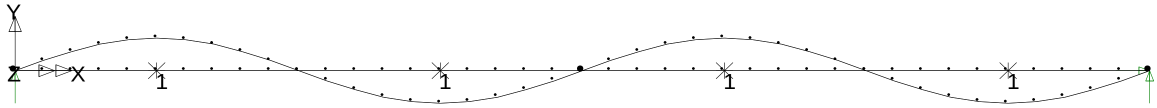
Continuing with the same structure as before but extended it to 2 spans with a 20 m total length. Support conditions are pinned at one end, simply supported at the other end and in the centre. The FE analysis gives the same fundamental frequency 0.5945 Hz as for the single span beam with a sinusoidal mode shape.



a. Displaced shape for load on end span



b. Displaced shape for load on centre span



c. First mode shape

Figure 3. Four-span simply supported beam

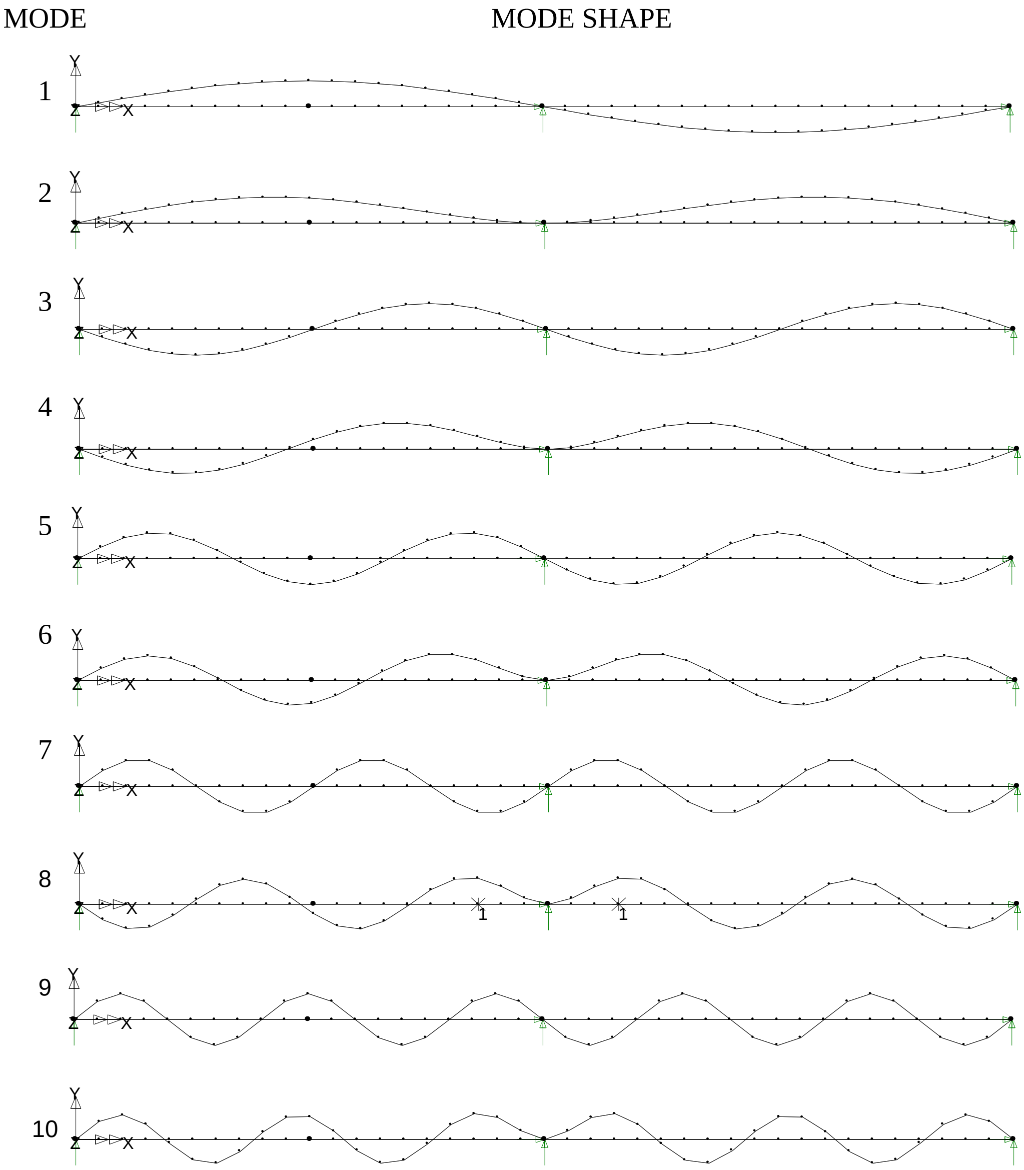


Figure 4. The mode shapes of two span beam

This, however, has twice the modal mass of the single beam and hence its stiffness is 144.22 N/m. Higher natural frequency modes occur at 0.929, 2.38, 3.01, 5.35 Hz.

For the case of unity load at the centre of one span the stiffness is 105.93 N/m, which is significantly smaller than the modal stiffness unlike the one span system. The displaced shapes for the point load and the mode shape are given in Figure 2. Again the normalised mode shape is divided by the modal stiffness. It can be seen that the similarities between mode shape and static displacement are now being lost.

FE Analysis of 4 span simply supported beam

Finally the analysis is repeated for a 4 span beam (40m). The analysis gives the same fundamental frequency of 0.5945 Hz and again a sinusoidal mode shape. The modal stiffness will be twice that of a 2 span system i.e. 288.44 N/m. Higher order modes get more complex with frequencies 0.694, 0.929, 1.999, 2.378 Hz. Note that the some natural frequencies are common between single and multi-span systems.

For the point load, consider two cases, one the centre of an end span and second the centre of a central span. These give point stiffnesses of 108.98 and 141.54 N/m respectively, both being much smaller than the fundamental modal stiffness. The displaced shapes for the two cases and the mode shape are given in Figure 3.

Key factors illustrated by the beam example

This beam model has been used to emphasise some important factors, the example being selected because the deflected shapes are simple to visualise. For the one span example, the deflected shapes for the mode shape and the point load are all somewhat similar. This is the reason why the static deflected shapes can be used as a substitution for the mode shape in many dynamic calculations, and also why the static displacement can be used to determine natural frequency.

However, this does not apply to multi-bay systems, because although the same natural frequencies may be encountered, the stiffness changes considerably with number of bays, and the mode shape and displacements due to static loads no longer correspond. Nevertheless, this does suggest that as part of structural testing, measurements of the point stiffness can supplement modal stiffness characteristics for system identification.

ON THE COMPOSITION OF STATIC DISPLACEMENTS

Single Span Beam

It is worth emphasising the links between static and dynamic displacement, as this is fundamental to this programme of work. Returning to the single span simply supported beam, which is the easiest system to visualise. The stiffness determined using the FE model was 76.16 N/m that equates to a central displacement of 0.01313 m under a unity point load.

This displacement could also have been determined using the formula

$$\frac{L^3}{48EI} = 0.0131332 \tag{6}$$

Alternatively this displacement could have been derived from the dynamic stiffness. Figure 4 shows the mode shapes for the first five modes of the simply supported beam, which are all sine waves with different wavelengths. Thus in all cases the modal value ϕ^2 equals 0.5. For a point load in the centre of the beam, there will be no displacement in the even numbered modes as the centre point is a node. For the odd numbered modes the central displacement due to a central load is the maximum modal value (ie unity). The static displacement is equivalent to the sum of the displacement of all of the modes.

The modal stiffness for each mode is 11.076 x 0.5 (2 x π x f)^2 hence the modal displacement is 1/(11.076 x 0.5 (2 x π x f)^2). The summation of the modal displacements for the odd modes is given in Table 1 along with modes considered.

From the table it can be seen that as the number of modes considered increases so the difference between total modal displacement and the static displacement decreases. The modal displacement for unit force is also known as flexibility. Hence the static displacement relates to the sum of the flexibility of each mode, but here it is dominated by the first mode. The effective difference between the flexibility of the first mode and the sum of the flexibilities of the remaining modes (which will be defined as the residual flexibility) is important in the developments given later in this project.

Table 1. The relationship between modal and static displacement

Modes considered	Total modal displacement	Static displacement (point load)
1	0.0129432	0.0131332
1+3	0.0131030	0.0131332
1+3+5	0.0131237	0.0131332
1+3+5+7	0.0131291	0.0131332
1+3+5+7+9	0.0131311	0.0131332
1+3+5+7+9+11	0.0131320	0.0131332
1+3+5+7+9+11+13	0.0131324	0.0131332
1+3+5+7+9+11+13+15	0.0131327	0.0131332

Table 2. The first ten modes of the two span beam

Mode number	Mode frequency (Hz)	Shape
1	0.5944	Sine wave – one central node (ie asymmetric about centre node)
2	0.9287	Approx. sine wave – one node – centre contra-flexure (ie symmetric about centre node)
3	2.3778	Sine wave – three nodes
4	3.0094	Approx. sine wave – three nodes – centre contra-flexure
5	5.3501	Sine wave – five nodes
6	6.2789	Approx. sine wave – five nodes – centre contra-flexure
7	9.5112	Sine wave – seven nodes
8	10.7373	Approx. sine wave – seven nodes – centre contra-flexure
9	14.8615	Sine wave – nine nodes
10	16.3850	Approx. sine wave – nine nodes – centre contra-flexure

Table 3. The relationship between modal and static displacement for various mode combinations for the two span beam

Modes considered	Total modal displacement	Static displacement
1	0.006481	0.009440
1+2	0.009136	0.009440
1+2+5	0.009216	0.009440
1+2+5+6	0.009274	0.009440
1+2+5+6+9	0.009284	0.009440
1+2+5+6+9+10	0.009293	0.009440

Two-Span Beam

Following the method given in the previous section, consider a 2 span beam of length 20m. This has twice the modal mass of the single span beam and hence its modal stiffness is 144.22 N/m, which equates to a central displacement of 0.006934 m under a unity modal load. For the case of unity load at the centre of one span the stiffness derived from the FE model is 105.93 N/m, which equates to a central displacement of 0.009440 m of the loaded span. Hence there is a much bigger difference between the static displacement and the fundamental modal displacement than for the single span beam. At the risk of being repetitive this will again be explained as it is central to some of the ideas developed later which will be concerned with measurement of static and dynamic stiffnesses.

Figure 4 shows the mode shapes of several modes. With the load position at the centre of one span, this will give a maximum modal load for some modes, but no modal load for those modes which have a node at this load position. Hence of the five modes shown, two can be neglected for this load case. In fact in this representation, several key modes have been omitted since these have a point of contra-flections at the centre support point, which complicates the simple calculation techniques given here. The first ten modes are given in Table 3 with a description of the mode shapes.

The five modes shown in the Figure, are modes 1, 3, 5, 7 and 9 in Table 2. Of these five modes, modes 3 and 7 have a nodal point at the load position, hence they have no response to the load. Modes 4 and 8 also have nodes at the load position.

The modal displacement is calculated using $1/(22.152 \times 0.5 (2 \times \pi \times f)^2)$, albeit this is only an approximation for modes 2, 6 and 10. The mass of 22.152 is twice that of the single span beam. The values for the displacement at the load position are given in Table 3.

Again it can be seen how the combination of modes builds up towards the static displacement, but note the significance of the modes derived from the FE analysis, especially the second mode in this 2 span example. The slight mis-match between modal and static displacement is due to the approximate calculations for modes 2, 6 and 10. It is also of interest to determine the displacement at the centre point of the other half of the beam. These are given in Table 4, noting that in this case modes, 2, 6 and 10 will be in anti-phase.

Discussion on items covered in sections of static displacements

The examples given above show how the static displacement is the summation of the modal displacements, and this is fundamental to structural analysis and, effectively, a different mathematical analysis of the same equations, i.e. the stiffness matrix for the structure, albeit the modal calculations include the mass of the structure. The significant point is that, for the first time, it is possible to take a relatively simple measurement of a point stiffness for a large structure, using the laser system. It is also possible to measure dynamic stiffness, using the methods developed at BRE over a number of years. The question is how can this information be used to help understand structural behaviour.

Table 5. The frequencies of the 8 lowest frequency modes of the various plates

	1	2	3	4	5	6	7	8
Plate 1	5.681	13.113	13.113	16.503	31.217	36.231	41.387	41.387
Plate 2	15.653	39.031	39.478	62.568	78.483	78.897	101.211	102.397
Plate 3	28.845	58.805	58.805	86.759	105.466	105.929	132.361	132.361
Plate 4	6.618	13.285	14.896	30.335	33.373	39.676	49.233	58.446
Plate 5	6.432	6.619	13.286	13.857	14.896	15.573	20.760	30.335
Plate 6	6.411	6.619	7.503	13.342	14.076	14.436	14.452	16.920
Plate 7	7.055	7.055	7.810	8.323	14.106	14.106	14.48	16.060
Plate 8	7.254	7.265	7.629	7.679	7.898	7.923	8.769	8.843

Table 6. Static and dynamic stiffnesses for various plates

Plate	'A' Point stiffness x 10^6 N/m	'B' Modal stiffness x 10^6 N/m	A/B
1. One bay + corner supports	21.28	24.81	0.858
2. One bay + simple supports	74.07	79.68	0.930
3. One bay + encastré supports	153.14	182.5	0.839
4. Two bay + plus Z restraints on the inner two columns	27.47	61.41	0.447

MODELLING OF PLATES / FLOORS

Having examined the behaviour of simple beams, the concepts will be expanded to include another dimension and plates (or floors) will now be considered. For the following calculations consider a floor of the concrete in-situ building at Cardington. The whole floor area was 4 by 3 bays based on supporting columns in a 7.5m x 7.5 m grid. The basic properties of the floors were:

Panel (bay) length and width	a = 7.5 m
Young's Modulus for concrete	E = 35.5 x 10^9
thickness of floor (plate)	h = 0.25
Mass per unit area	γ = 2400 x 0.25
Poisson's ratio	υ = 0.2
Number of half waves along horizontal axis	i = 1
Number of half waves along vertical axis	j = 1

(i = j = 1 defines the fundamental mode)

On this floor the lowest measured frequency was 8.54Hz.
Measured frequencies of the central bays were 11.96Hz

Consider a 7.5 x 7.5 m plate with various support arrangements. The frequency determined using the Cardington data is evaluated using theoretical formula [2].

The natural frequency of plate with corner supports:

$$f = \frac{7.12}{2\pi a^2}\sqrt{\frac{Eh^3}{12\gamma(1-\upsilon^2)}} \tag{7}$$

f = 5.707 Hz

The natural frequency of plate with simple supports

$$\lambda = \sqrt{\pi^2\left[i^2 + j^2\left(\frac{a}{b}\right)^2\right]^2} \tag{8}$$

λ = 4.443

$$f = \frac{\lambda^2}{2\pi a^2}\sqrt{\frac{Eh^3}{12\gamma(1-\upsilon^2)}} \tag{9}$$

f = 15.822 Hz

The natural frequency of equivalent fully clamped plate is

$$f = \frac{35.99}{2\pi a^2}\sqrt{\frac{Eh^3}{12\gamma(1-\upsilon^2)}} \tag{10}$$

f = 28.847 Hz

It can be seen that the lowest measured frequency (8.54 Hz) is between that of a plate with corner supports and a simple supported plate. This is not unreasonable given the fact that the measured frequency is mainly motion of a corner section of the floor which may be considered to have four corner supports plus one simply supported edge (where it is attached to the rest of the floor area). The centre bays have predominant frequencies of 11.96 Hz, which is nearer to the frequency of the simply supported plate.

However, the main point to be gained from these simple examples is the importance of the boundary conditions, which have an enormous influence on the frequencies (and stiffness) and hence the floors response to dynamic loads. It will be shown later that the boundary conditions also have an enormous influence on strength. These are just simple plates; the situation becomes more complicated for the multi-bay flooring systems commonly used for modern office floors.

FE analysis of plates

Following a similar approach to that adopted with the simple beam, consider an FE model of the same floor. Here the FEA program LUSAS is used and in this example thin plate elements are used in the modelling, with the plate being divided into 100 elements. The following plates were considered.

Plate 1, One bay plate (7.5. x 7.5 m) corner supports only, one pinned, one YZ restraint, one XZ restraint, one Z restraint.

Plate 2, One bay, corner supports, & 4 sides simply supported.

Plate 3, One bay, corner supports & 4 sides encastré

Plate 4. One bay, corner supports + one side simply supported.

Plate 5. Two bay plate (15m x 7.5m) corner supports plus Z restraints on the inner two columns

Plate 6. Three x one bay plate (22.5m x 7.5m) corner supports plus intermediate Z restraints

Plate 7. Two x two bay plate (15m x 7.5m) corner supports plus intermediate Z restraints

Plate 8. Four x three bay plate (30m x 22.5m) corner supports plus intermediate Z restraints

The frequencies were derived for the 8 lowest frequency modes in each example and these are given in Table 5.

From Table 5 it can be appreciated that the fundamental modes of the single bay plates align with the calculations using the simple formula. The multi-bay examples show far more complex behaviour, with 'bands' of modes being present, ie the single bay has a distinct fundamental frequency, the two bay system has a second mode at a frequency close to the fundamental, the three bay system has three modes in this band etc.

STATIC AND MODAL STIFFNESSES OF PLATES

Following the development adopted with the beam example, consider both static and dynamic stiffnesses of some of the above examples.

Plate 1, One bay plate with corner supports

Static point stiffness is 21.28 x 10^6, UDL stiffness is 33.22 x 10^6

From FE modelling ϕ^2= 0.577, hence the modal mass of the plate/floor is 7.5 x 7.5 x 2400 x 0.25 x 0.577 = 19473.75. Thus given the frequency of 5.681 Hz, the modal stiffness is 19473.75 x (2 x π x 5.681)² = 24.81 x 10^6 N/m,

Plate 2, One bay plate with corner supports + simple supports on 4 sides

Static point stiffness is 74.07 x 10^6, UDL stiffness is 241.13 x 10^6

With the simply supported case the mode shape is well defined hence ϕ^2

$$\varphi^2 = \int_0^1\int_0^1 \sin^2 \pi x . \sin^2 \pi y . dy . dx = 0.25 \qquad (11)$$

Hence the modal mass of the plate/floor is 7.5 x 7.5 x 2400 x 0.25 x 0.25 = 8237.5. Thus, given the frequency of 15.653 Hz, the modal stiffness is 8237.5 x (2 x π x 15.653)² =79.68 x 10^6 N/m,

From the FE model, and quick evaluation of the modal parameter, ϕ^2= 0.2534

Therefore this suggests a similar relationship between static and modal stiffness as seen for the simple beam.

Plate 3, One bay plate with corner supports + encastré supports on 4 sides

Static point stiffness is 153.14 x 10^6, UDL stiffness is 675.68 x 10^6

From FE modelling ϕ^2=0.1646, hence the modal mass of the plate/floor is 7.5 x 7.5 x 2400 x 0.25 x 0.1646 = 5555.25. Thus given the frequency of 28.845Hz, the modal stiffness is 5555.25x (2 x π x 28.845)² = 182.5 x 10^6 N/m,

Plate 5 Two bay plate corner supports plus Z restraints on the inner two columns

Here is a two panel system

For the first mode the frequency is 6.432 Hz and ϕ^2=0.557

For the second mode the frequency is 6.619 Hz and ϕ^2= 0.541

As with the beam, these are acting in the same direction on one half of the plate but in opposite directions on the other half of the plate.

Evaluating stiffness gives

Mode 1 k1 = 2 x 7.5 x 7.5 x 2400 x 0.25 x 0.557 x (2 x π x 6.432)² = 61.41 x 10^6 N/m,

Mode 2 k2 = 2 x 7.5 x 7.5 x 2400 x 0.25 x 0.594 x (2 x π x 6.619)² = 69.35 x 10^6 N/m,

Suppose a unit load were applied at the centre of one of the bays. From the above, the displacement where the load was applied would be approximately

1/k1+1/k2 = (1.628 + 1.442) x 10^{-8} = 3.070 x 10^{-8}

And the displacement in the centre of the panel where the load was not applied would be approximately

1/k1-1/k2 = (1.628 - 1.442) x 10^{-8} = 0.186 x 10^{-8}

Calculating the same values with the FE program gives 3.64 x 10^{-8} and 0.101 x 10^{-8}. These are not exactly the same as the above values; hence some higher modes would have to be considered for a better correlation. The point stiffness at the load position is the inverse of the displacement that equals 27.47 x 10^6 N/m. The static point and modal stiffnesses for the four plates are summarised in Table 6.

From Table 6 it can be appreciated that although the point static stiffness and modal stiffness are similar for the single bay system, whatever the boundary conditions, for multi-bay system this similitude disappears. This is a similar concept to that demonstrated with the simple beam.

CONCLUSIONS

The analysis of simply supported beam are as follows:

The number of modes considered increases the difference between total modal displacement and the static displacement decreases.

The first mode dominated the sum of the modal displacement.

The analysis that has been undertaken on the plates illustrates two key points.

The boundary conditions for the plates have a significant effect on the structural stiffness and frequencies. The static and modal stiffnesses are affected equally.
The number of bays, does not affect the fundamental frequency significantly, although it does produces bands of modes. This is reflected in large differences between static and dynamic stiffness.

It is therefore apparent that measurements of both static and dynamic stiffness can both provide useful information on the elastic behaviour of a structure.

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