The Use of Dirac Oscillator as Medium Substrate for Quantum Heat Engine

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Abstract: The research on Dirac oscillators has been increasing recently. In this paper, a quantum heat engine by means of a Dirac oscillator that interacts with the external magnetic field is proposed. Relativistic Landau energy levels are used to perform the iso-energetic cycle. The large magnetic field is set to obtain a perfect thermodynamic cycle. Some stable performances at a certain magnitude of magnetic field and expansion parameter range are obtained. When the value of efficiency is compared with the non-relativistic case, an opposite result occurs. Therefore, a quantum heat engine using a Dirac oscillator doesn't govern like a classical oscillator.

Keywords: Dirac Oscillator; Quantum Heat Engine; Iso-energetic Cycle

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I. INTRODUCTION

The quantum heat engine has been progressing lately, along with advances in nanotechnology. A wide range of medium substrates for quantum heat engines has been proposed, such as a one-dimensional potential well [1–5], a harmonic oscillator [6–8], etc. There have also been studied quantum heat engines with minimal length [9]. For one-dimensional potential well, the efficiency of non-relativistic and relativistic quantum heat engines have been compared. It was proved that the relativistic quantum heat engine in a potential well has a lower efficiency than the non-relativistic case [10].

A harmonic oscillator as a medium substrate without an external magnetic field has been carried out by utilizing the master equation [7]. Also, a quantum heat engine in non-relativistic cases has been studied using an external magnetic field which is analogous to the width of the potential well [11]. The thermodynamics processes are governed by a one-dimensional potential well quantum heat engine. The applied medium substrate is a single particle in a quantum dot semiconductor cylinder [12].

This paper discusses the quantum heat engine by means of a Dirac oscillator as a medium substrate. It is assumed that the spin on the Dirac oscillator will be ignored. The magnetic field is converted during the process so that the system undergoes a transition state. The magnetic field is set to obtain a perfect cycle. These models are expected to make a valuable contribution to quantum dot semiconductor applications.

II. DIRAC OSCILLATOR

A Dirac oscillator is the relativistic version of the quantum harmonic oscillator. There are several topics of Dirac oscillator in the case of (2+1) dimensions [13–15]. In this paper, the Dirac oscillator using an external magnetic field as a ref [13] is used. The Hamiltonian of the system is given by

$$H = c\boldsymbol{\alpha}\cdot\left(p - e\boldsymbol{A}/c - im\omega\beta\boldsymbol{r}\right) + \beta mc^2, \tag{1}$$

with $\mathbf{A}$ is the external magnetic field vector potential, e is electron charge that acts as the Dirac oscillator. In this model, the (2+1)-oscillator dimensions are considered, with the external magnetic field on the z-axis direction so that the vector potential is $\mathbf{A} = \left(-\frac{B}{2}y, \frac{B}{2}x, 0\right)$. Because of the two spatial dimensions, Pauli matrices become $\alpha_x = \sigma_x$, $\alpha_y = \sigma_y$, and $\beta = \sigma_z$. In order to get an analytic solution, mapping as ref. [13] is carried out, in order to obtain a modified Hamiltonian as follows

$$H = c\boldsymbol{\alpha}\cdot\left(\boldsymbol{p} - im\widetilde{\omega}\beta\boldsymbol{r}\right) + \beta mc^2, \tag{2}$$

with $\tilde{\omega} = \omega - |e|B/2mc$. From the results of completion of the Dirac equation $H\Psi = E\Psi$, relativistic Landau energy levels are obtained as follows

$$E_n^{\pm} = \pm mc^2\sqrt{1 + \frac{4\hbar\widetilde{\omega}}{mc^2}n},\; n = 1, 2, 3, .. \tag{3}$$

III. SINGLE PARTICLE QUANTUM HEAT ENGINE

According to Eq. (3), the energy levels of the system depend on the value of an external magnetic field. Then the energy of a single particle system is

$$E = \left\langle \hat{H} \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{H}\right) = \sum_n p_n(B)E_n(B). \tag{4}$$

By differentiation, it shows

$$dE = \sum_n p_n(B)dE_n(B) + E_n(B)dp_n(B). \tag{5}$$

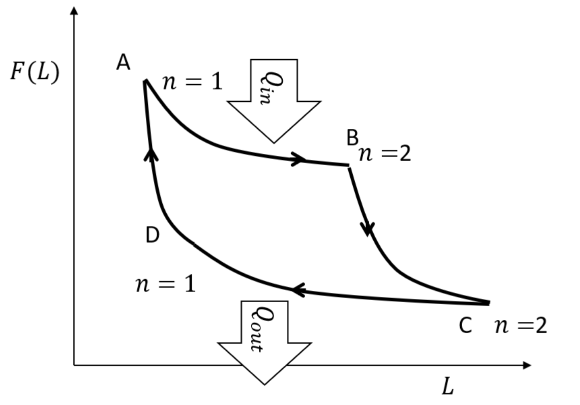

FIG. 1: Iso-energetic cycle. B represents an external magnetic field that is applied to the system, and M is the magnetization of the medium substrate

The energy equation above is analogous to the First Law of Thermodynamics.

$$dU = dQ - dW. \tag{6}$$

Because the probabilities of each state of the system is associated with entropy (von Neumann entropy).

$$S = -k_B \sum_n p_n(B) \ln p_n(B). \tag{7}$$

Then, the change in the total energy of the system is analogous to the change of internal energy in classical thermodynamics. On the right side, the first term of Eq. (5) is analogous to the work, and the second term is analogous to the heat exchange. The external magnetic field is related to the internal characteristics of the system, as well as the width of the potential well of the one-dimensional quantum heat engine. Thus, the change of the magnetic field does not provide an addition of heat to the system. During the iso-energetic process, the total energy is unchanged. An external magnetic field is set in a way that allows the system state changes from $n = 1$ to $n = 2$. During the process, all incoming heat is converted into work. Whereas during the adiabatic process, the magnetic field remains changing although there is no heat intake. The running of the magnetic field will change the energy level of each state as compensation for the work that has been done to the system.

IV. ISO-ENERGETIC PROCESSES

The iso-energetic cycle has two iso-energetic processes and two iso-entropic processes, as shown in Fig. 1. First, the iso-energetic AB is calculated in which the heat penetrating into the system changes completely into work. In this model, the initial state of the system is a ground state $n = 1$ with a constant magnetic field B_A. During the iso-energetic process, the magnetic field changes. At the end of the process, the system is fully in the state $n = 2$. During the process, the probability value and the total energy are changed as follows

$$p_1(B) + p_2(B) = 1, \tag{8}$$

$$p_1(B)E_1(B) + p_2(B)E_2(B) = E_1(B_A). \tag{9}$$

Based on the ref. [2], the value of the incoming heat is given by

$$Q_{AB} = \sum_n \int_{B_A}^{B_B} E_n(B) \frac{dp_n(B)}{dB} dB. \tag{10}$$

Using Eqs. (8) and (9), the heat value can be written as follows

$$Q_{AB} = \int_{B_A}^{B_B} [E_1(B) - E_2(B)] \frac{d}{dB} \left[\frac{E_1(B_A) - E_2(B)}{E_1(B) - E_2(B)} \right] dB. \tag{11}$$

Using Landau relativistic energy level Eq. (3), it shows

$$\begin{aligned} Q_{AB} = -mc^2 \Bigg[&\tanh^{-1}\left(\frac{E_2(B_B)}{mc^2}\right) + \tanh^{-1}\left(\frac{E_1(B_B)}{mc^2}\right) \\ &- \tanh^{-1}\left(\frac{E_2(B_A)}{mc^2}\right) - \tanh^{-1}\left(\frac{E_1(B_A)}{mc^2}\right)\Bigg]. \\ &- E_1(B_A) \ln\left[\frac{eB_B - 2m\omega c}{eB_A - 2m\omega c}\right] + \frac{1}{2} E_1(B_A) \ln\left[\frac{F(B_B)}{F(B_A)}\right] \end{aligned} \tag{12}$$

with

$$F(B) = 3ecB\hbar - m^2c^4 - 6mc^2\hbar\omega - E_1(B)E_2(B)$$

During process AB, the total energy of the system remains constant. Thus, the magnetic field at the end of the process AB satisfies

$$B_B = \frac{1}{2} B_A + \frac{m\omega c}{e}. \tag{13}$$

The value of the magnetic field is not the same as the non-relativistic case, as in ref. [11]. In this paper, the magnitude of the magnetic field at the end of the process depends on the mass m, charge e, and frequency ω. At the same time, the value ω will be determined by geometric scale $\ell_d = \sqrt{\hbar/m\omega}$. In the relativistic case, in order to obtain the fully final state $n = 2$, the magnitude of the magnetic field is also determined by these three parameters.

The following process is the iso-entropic process BC. During this process, there is no heat absorbed into the system. The work done by the system depends on changes in the internal energy. There is no state change, so the system is entirely on the state $n = 2$. Thus, during process BC, the work is given by

$$W_{BC} = E_2(B_C) - E_2(B_B). \tag{14}$$

By substituting Landau relativistic energy levels, it becomes

$$\begin{aligned} W_{BC} = mc^2 &\sqrt{1 + \frac{8\hbar}{mc^2}\left(\omega - \frac{eB_C}{2mc}\right)} \\ &- mc^2 \sqrt{1 + \frac{8\hbar}{mc^2}\left(\omega - \frac{eB_B}{2mc}\right)}. \end{aligned} \tag{15}$$

During the iso-entropy process, there is a change in the magnitude of the magnetic field from B_B to B_C. In the ref. [11], the iso-entropic process involves an expansion parameter. In order to make a good analogy, the geometric scale by an external magnetic field is defined as follows

$$\ell_B = \sqrt{\frac{\hbar}{m\omega_B}}, \tag{16}$$

with

$$\omega_B = \frac{eB}{m}. \tag{17}$$

So that the expansion parameter can be involved with

$$\frac{\ell_{B_C}}{\ell_{B_B}} = \alpha_1. \tag{18}$$

The expansion parameter α_1 is not written in a general form. The reason is the expansion parameter of the next iso-entropic process is not necessarily the same. Using Eqs. (16) and (17) show

$$B_C = \frac{1}{\alpha_1^2} B_B. \tag{19}$$

Then the cycle continues to the iso-energetic process CD. In this process, the state of the system changes from $n = 2$ state to $n = 1$ state again. Because the total energy remains constant, the process is fulfilled by

$$E_2(B_C) = E_1(B_D). \tag{20}$$

Thus obtained the relation

$$B_D = 2B_C - \frac{2m\omega c}{e}. \tag{21}$$

While the heat that comes out can be calculated by the integral as follows

$$Q_{CD} = \int_{B_C}^{B_D} \left[E_1(B) - E_2(B)\right] \frac{d}{dB}\left[\frac{E_2(B_C) - E_2(B)}{E_1(B) - E_2(B)}\right] dB, \tag{22}$$

resulting

$$\begin{aligned} Q_{CD} = &-mc^2 \left[\tanh^{-1}\left(\frac{E_1(B_D)}{mc^2}\right) + \tanh^{-1}\left(\frac{E_2(B_D)}{mc^2}\right) \right. \\ &\left. - \tanh^{-1}\left(\frac{E_1(B_C)}{mc^2}\right) - \tanh^{-1}\left(\frac{E_2(B_C)}{mc^2}\right)\right] \\ &- E_2(B_C) \ln\left[\frac{eB_D - 2m\omega c}{eB_D - 2m\omega c}\right] + \frac{1}{2} E_2(B_C) \ln\left[\frac{F(B_D)}{F(B_C)}\right]. \end{aligned} \tag{23}$$

And the later process is the iso-entropic process, in which the end of the process is returning to the initial state. The work during this process is

$$\begin{aligned} W_{DA} = &\, mc^2 \sqrt{1 + \frac{4\hbar}{mc^2}\left(\omega - \frac{eB_A}{2mc}\right)} \\ &- mc^2 \sqrt{1 + \frac{4\hbar}{mc^2}\left(\omega - \frac{eB_D}{2mc}\right)}, \end{aligned} \tag{24}$$

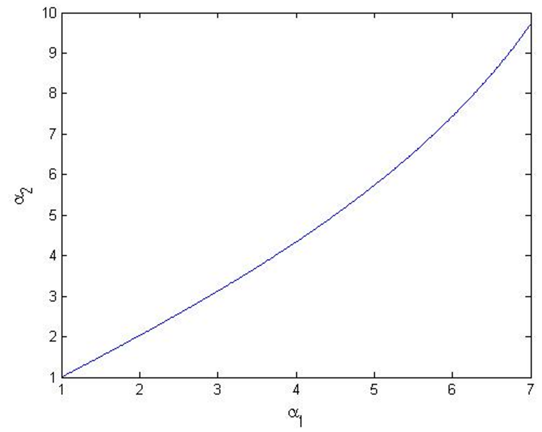

FIG. 2: Relation between α_1 and α_2.

with the geometry scale satisfies

$$\frac{\ell_{B_D}}{\ell_{B_A}} = \alpha_2. \tag{25}$$

the relationship of the initial and the final magnetic field is

$$B_D = \frac{1}{\alpha_2^2} B_A. \tag{26}$$

In order to obtain a reversible cycle, the magnitude of the magnetic field must satisfy the following cycle

$$B_A \xrightarrow[\frac{1}{2}B_A + \frac{m\omega c}{e}]{} B_B \xrightarrow[\frac{1}{\alpha_1^2}B_B]{} B_C \xrightarrow[2B_C - \frac{2m\omega c}{e}]{} B_D \xrightarrow[\alpha_2^2 B_D]{} B_A. \tag{27}$$

The first and second expansion parameter could be related as follows

$$\frac{\left(1 - \frac{\alpha_2^2}{\alpha_1^2}\right)}{2\left(\frac{\alpha_2^2}{\alpha_1^2} - \alpha_2^2\right)} = \frac{m\omega c}{eB_A}. \tag{28}$$

It can be illustrated by Fig. 2. According to Fig. 2, the value of the first expansion parameter α_1 is not necessarily the same as the second α_2. Then the efficiency of the machine can be obtained as

$$\eta = 1 - \frac{Q_{CD}}{Q_{AB}}, \tag{29}$$

substituting Eqs. (12), (13),(21), and (23) related as follows

$$\eta = 1 - \frac{\left[\tanh^{-1}\Theta(4a_C) - \tanh^{-1}\Theta(a_C)\right] - \frac{1}{2}\Theta(2a_C)\ln G(a_C)}{\left[\tanh^{-1}\Theta(a_A/2) - \tanh^{-1}\Theta(2a_A)\right] + \frac{1}{2}\Theta(a_A)\ln G(a_A)}, \tag{30}$$

we define

$$G(a) = \left[\frac{\frac{3}{4}a - \frac{1}{4} - \frac{1}{4}\Theta(a)\Theta(4a)}{\frac{3}{2}a - 1 - \Theta(a)\Theta(4a)}\right]$$

with $\Theta(a) = \sqrt{1+a}$ and $a_A = \frac{4\hbar}{mc^2}\left(\omega - \frac{eB_A}{2m}\right)$, $a_C = \frac{4\hbar}{mc^2}\left(\omega - \frac{eB_C}{2m}\right)$. Variable a_A and a_C are related by $a_C =$

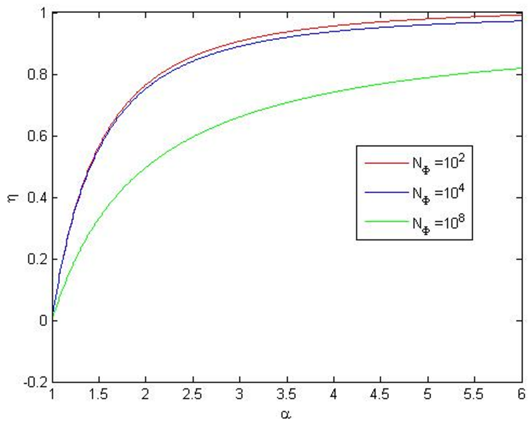

FIG. 3: Relation between expansion parameter with the efficiency by initial magnetic field with various B_A. We choose $\ell_d = 10^{-7}\ m$.

$\frac{1}{2\alpha_1^2} a_A + \frac{4\hbar}{mc^2}\omega\left(1 - \frac{1}{\alpha_A^2}\right)$. We define N_Φ as a quantity that expressed the value of initial magnetic field B_A as $B_A = \frac{2m\omega}{e} N_\Phi$. Due to relativistic property, not all values of N_Φ producing all cyclic processes. It can be said initial value of the magnetic field is large enough and has a minor change during the iso-energetic cycle. From Fig. 3, it appears that the greater the magnetic field, the smaller the engine efficiency. That differs from the ref [11], which provides the opposite conclusion.

V. DISCUSSION AND CONCLUSION

A Quantum heat engine with the Dirac oscillator as the substrate medium produces engine efficiency values through an isoenergetic cycle. In the non-relativistic case, the efficiency is proportional to the initial value of the magnetic field. In a relativistic case, this happens inversely. The larger the initial magnetic field, the smaller the efficiency value. Moreover, what should be noted is that not all initial magnetic field values produce cyclic cycles in a quantum heat engine using the Dirac oscillator as the substrate medium. The expansion parameter's value also affects the engine's efficiency, as in the non-relativistic case.

VI. ACKNOWLEDGMENT

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[1] C. M. Bender, D. C. Brody, and B. K. Meister, "Quantum mechanical carnot engine," *Journal of Physics A: Mathematical and General*, vol. 33, no. 24, p. 4427, 2000.

[2] H.-T. Quan, Y.-x. Liu, C.-P. Sun, and F. Nori, "Quantum thermodynamic cycles and quantum heat engines," *Physical Review E*, vol. 76, no. 3, p. 031105, 2007.

[3] C. M. Bender, D. C. Brody, and B. K. Meister, "Entropy and temperature of a quantum carnot engine," *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, vol. 458, no. 2022, pp. 1519–1526, 2002.

[4] J. Wang and J. He, "Optimization on a three-level heat engine working with two noninteracting fermions in a one-dimensional box trap," *Journal of Applied Physics*, vol. 111, no. 4, p. 043505, 2012.

[5] E. Latifah and A. Purwanto, "Multiple-state quantum carnot engine," *Journal of Modern Physics*, vol. 2011, 2011.

[6] B. Lin and J. Chen, "Performance analysis of an irreversible quantum heat engine working with harmonic oscillators," *Physical Review E*, vol. 67, no. 4, p. 046105, 2003.

[7] Y. Rezek and R. Kosloff, "Irreversible performance of a quantum harmonic heat engine," *New Journal of Physics*, vol. 8, no. 5, p. 83, 2006.

[8] J. Arnaud, L. Chusseau, and F. Philippe, "Carnot cycle for an oscillator," *European journal of physics*, vol. 23, no. 5, p. 489, 2002.

[9] A. Purwanto, H. Sukamto, B. Subagyo, *et al.*, "Quantum carnot heat engine efficiency with minimal length," *Journal of Modern Physics*, vol. 6, no. 15, p. 2297, 2015.

[10] E. Munoz and F. J. Pena, "Quantum heat engine in the relativistic limit: The case of a dirac particle," *Physical Review E*, vol. 86, no. 6, p. 061108, 2012.

[11] E. Munoz and F. J. Pena, "Magnetically driven quantum heat engine," *Physical Review E*, vol. 89, no. 5, p. 052107, 2014.

[12] D. Englund, D. Fattal, E. Waks, G. Solomon, B. Zhang, T. Nakaoka, Y. Arakawa, Y. Yamamoto, and J. Vučković, "Controlling the spontaneous emission rate of single quantum dots in a two-dimensional photonic crystal," *Physical review letters*, vol. 95, no. 1, p. 013904, 2005.

[13] B. Mandal and S. Verma, "Phys. lett. a," 2010.

[14] B. P. Mandal and S. K. Rai, "Noncommutative dirac oscillator in an external magnetic field," *Physics Letters A*, vol. 376, no. 36, pp. 2467–2470, 2012.

[15] D. Nath and P. Roy, "Dirac oscillator in perpendicular magnetic and transverse electric fields," *Annals of Physics*, vol. 351, pp. 13–21, 2014.