History-like Path of Identical Particle Systems in Topological Space M

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Abstract

In this paper, a new concept has been introduced related to the construction of inequivalent quantizations for a system, which is called "history-like path" for particles. This is a collection of non-causal homotopical paths. Each homotopy class of paths will be labelled with a "word" constructed from the generators of fundamental group $\pi_1(Q_N(\Sigma_i))$ of a configuration space of the system in a space-like Σ_i which is a deformation retract of M_i region that is a sub-space-time of M in which there is no singular slice (a slice M_c in M that contains a singular point). The labels determine the generators and their relations in constructing the fundamental group $\pi_1(Q_N)$ of the system.

KEYWORDS: homotopical path, fundamental group, topological space, and topology change

I. INTRODUCTION

The topology of the spatial space which accommodates N-identical particles will affect the allowed quantizations for the system[1–3]. Different topology may lead to a different configuration space for the system. Inequivalent quantizations are in (1-1)-correspondence with the irreducible unitary representations (IUR's) of the fundamental group $p_1(Q)$ for the configuration space of the system. These many possibilities in quantizing the system in turn may result in many possibilities for the statistics of the system [4, 5].

The quantization process above has to be done in a fixed spatial topology framework. The process meet serious difficulties when it is faced with a theoretical fact that in the quantum gravity and the cosmology theory, or instead in the elementary quantum theory, any spatial topology is changeable[6–10]. Related to this problem, the interesting case for investigation is how to describe the quantum states involving any spatial topology choice and change. In this case, the canonical quantization procedure is no longer valid. It is related to the fact that the canonical quantizations of identical particle systems are formulated in a fixed spatial topological background.

To handle this problem, we need a new concept related to the homotopical path of particles that can accommodate any spatial topology change. As a continuation of an initial investigation has been introduced in our paper[11], and elaboration of the concept based on the result reported in our paper has been carried out[12].

The main problem is how to construct and to label homotopical paths of identical particles such that the procedure of the quantizations could still be done although its spatial topology changes. A homotopical path, called as "history-like path", is a non-causal evolution path in M whose final point is identified with its initial point in M.

II. BASIC CONCEPT AND MEANING

When there is no a spatial topology change, a space-time can be written as $M = \Sigma \times I$ where the topology of Σ is fixed; $\Sigma_{\mathcal{I}}$ and $\Sigma_{\mathcal{F}}$ are mutually homeomorphic. Furthermore, all of the slices M_t's in M are mutually homeomorphic. Thus, each point on each M_t in M can be identified with each point on, say that $\Sigma_{\mathcal{I}}$. Therefore, one can construct the history-like paths of particles in M that emanate from $\Sigma_{\mathcal{I}}$ and enter into $\Sigma_{\mathcal{F}}$.

Consider 2 identical particle in $M = \mathbb{R}^2_\circ \times I$ ($\mathbb{R}^2_\circ$ is $\mathbb{R}^2$ with one puncture). All of possible paths can be projected into one of the M_t slices in M, for example, into $\Sigma_{\mathcal{I}}$. Therefore, there is a bijective map from the set of all history-like paths to the set of elements of $\pi_1(\Sigma_{\mathcal{I}})$.

When a spatial topology change occurs, $\Sigma_{\mathcal{I}}$ and $\Sigma_{\mathcal{F}}$ of M are no longer mutually homeomorphic. As a consequence, the points on the two boundary regions can not always be identified. To handle this problem, one assume that in each M_t, for all different t, there exist a local region U_t, which is homeomorphic to all the local region $U_{t'}$ in other $M_{t'}$ such that identification of points on these different regions can be done. Moreover, *for a set of points of N-particles $\{y_1, ...y_N\}$ on the $\Sigma_{\mathcal{F}}$ there exist a neighborhood U_y of $\{y\}$ homeomorphic with U_x neighborhood of $\{x\}$ on the $\Sigma_{\mathcal{I}}$. The x's and y's are initial and final points of history-like paths, respectively.*

In fact, any M can be divided into many regions not containing a singular slice $\mathcal{M}_i = (\mathcal{M}_1 \equiv M_{(0,c_1-\epsilon)}, ..., \mathcal{M}_{m+1} \equiv M_{(c_m+\epsilon,1)})$ where $i = m+1$ (m is a counting how many spatial topology change was happened) and ϵ is an infinitesimal positive real number, and many regions containing a singular slice M_{c_j}, $M_{(c_j-\epsilon,c_j+\epsilon)}$ where $j = 1, 2, ..., m$.

The history-like paths in $\mathcal{M}_i$ can be labeled as in the case

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when there is no topology change. In the $M_{(c_j-\epsilon,c_j+\epsilon)}$ regions, it can be seen that any history-like path can be homotopically deformed so that it will occupy $\mathcal{M}_{j-1}$ or $\mathcal{M}_{j+1}$ (but may be only one of these regions). This can be done as long as $M_{(c_j-\epsilon,c_j+\epsilon)}$ path connected. Thus, all of the history-like paths can be labeled as follow: *Any history-like path of an identical particle system in the $(d+1)$-dimensional space-time M with m-times topology changes is given by a composition of history-like paths of each $\mathcal{M}_i \subset M$ ($i = m+1$):*

$$h = h_{m+1} \circ h_m \circ \cdots \circ h_1. \qquad (1)$$

Equivalent classes of history-like paths $[h_i]$ in $\mathcal{M}_i$, which are labeled by "words" W in the group theory of meaning, are in a (1-1)-correspondence with the elements of $\pi_1(Q(\Sigma_i))$, where Σ_i is a d-dimensional spatial space which is a deformation retract of $\mathcal{M}_i = \Sigma_i \times I$.

III. THE CASES WITH $(2+1)$-DIMENSIONAL M

Consider several simple cases corresponding to three kinds of topology change(s) described as follow:

(a1) $\mathbb{R}^2$ changes into $\mathbb{R}^2$ with one puncture ($\mathbb{R}^2 \Rightarrow \mathbb{R}^2_\circ$);

(a2) $\mathbb{R}^2$ with one puncture changes into $\mathbb{R}^2$ with two punctures ($\mathbb{R}^2_\circ \Rightarrow \mathbb{R}^2_{\circ\circ}$); a puncture split into two punctures ($\mathbb{R}^2_\circ \Rightarrow \mathbb{R}^2_{\circ\circ}$);

(a3) $\mathbb{R}^2$ changes into $\mathbb{R}^2$ with one puncture and then changes into $\mathbb{R}^2$ with two punctures; two punctures emerge from which a puncture split ($\mathbb{R}^2 \Rightarrow \mathbb{R}^2_\circ \Rightarrow \mathbb{R}^2_{\circ\circ}$).

Consider a particle in (a1). Several history-like paths of the system, for example, are shown in Fig. 1 corresponding to

$$h_a = h_2 \circ h_1 = [\alpha] \circ [\mathbf{1}], \qquad (2a)$$
$$h'_a = h_2 \circ h_1 = [\mathbf{1}] \circ [\mathbf{1}]. \qquad (2b)$$

The path h_a can be deformed into h'_a. Thus, there is a redundancy corresponding to the labels of these paths. Since these paths are equivalent, one can choose the simplest label (h'_a). So, for N identical particles in (a1), $W(\alpha, \sigma_i) \sim W(\sigma_i)$, and

$$[h]_{(\mathrm{a1})} = \{W(\sigma_i) \mid 1 \leqslant i \leqslant N-1\}. \qquad (3)$$

Next, consider a particle in (a2). From Fig.2, it can be seen that all of the history-like paths in $\mathcal{M}_1$ can be deformed into trivial paths that they can be labeled as $h = h_2 \circ [\mathbf{1}]$,

$$h_a = [\alpha][\beta] \circ [\mathbf{1}], \qquad (4a)$$
$$h_b = [\alpha] \circ [\mathbf{1}], \qquad (4b)$$
$$h_c = [\beta] \circ [\mathbf{1}], \qquad (4c)$$
$$h_d = [\alpha][\beta][\alpha^{-1}] \circ [\mathbf{1}]. \qquad (4d)$$

For N identical particles, $W(\alpha, \beta, \gamma', \sigma_i) \sim W(\alpha, \beta, \sigma_i)$, thus

$$[h]_{(\mathrm{a2})} = \{W(\alpha, \beta, \sigma_i) \mid 1 \leqslant i \leqslant N-1\}. \qquad (5)$$

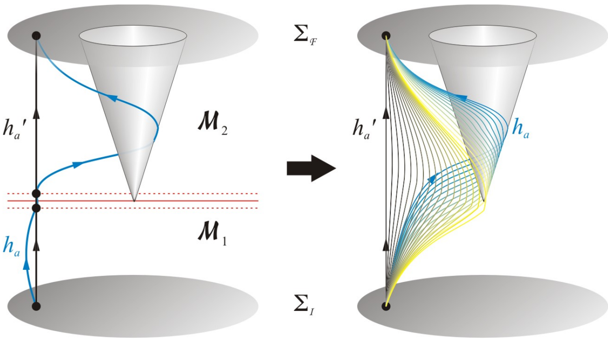

FIG. 1: $h_a = [\alpha] \circ [\mathbf{1}]$ and $h'_a = [\mathbf{1}] \circ [\mathbf{1}]$ of a particle in (a1). All paths around the puncture can be deformed into a trivial path (right figure)

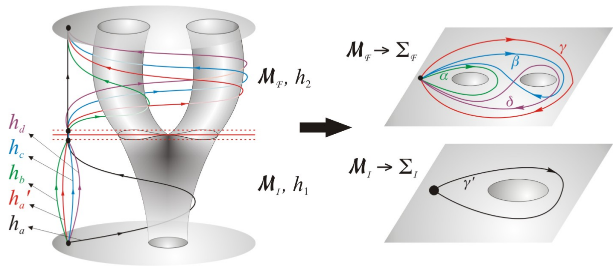

FIG. 2: The possible history-like paths of a particle in (a2)

Finally, consider a particle in (a3). M can be divided into $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$. From Fig.3, $h_a \sim h'_a \sim h''_a$. The path $h'_c = [\beta^{-1}][\alpha^{-1}][\beta][\alpha][\beta^{-1}][\alpha^{-1}] \circ [\mathbf{1}] \circ [\mathbf{1}]$ with its first term allowed to be rewritten as $[\gamma(2)^{-1}][\beta][\alpha][\gamma(1)^{-1}]$, where $[\gamma] = [\alpha][\beta]$ and the $(1), (2)$ labels express the path order. From the last figure of Fig. 3, $[\gamma(1)^{-1}]$ can be deformed into $[\mathbf{1}]$, but $[\gamma(2)^{-1}]$ can not, because it is restricted by the previous path ($[\beta][\alpha]$). Therefore, $h'_c \sim h_c$. Thus,

$$h_a = [\mathbf{1}] \circ [\mathbf{1}] \circ [\mathbf{1}], \qquad (6a)$$
$$h_b = [\alpha] \circ [\mathbf{1}] \circ [\mathbf{1}], \qquad (6b)$$
$$h_c = [\beta^{-1}][\alpha^{-1}][\beta][\alpha] \circ [\mathbf{1}] \circ [\mathbf{1}]. \qquad (6c)$$

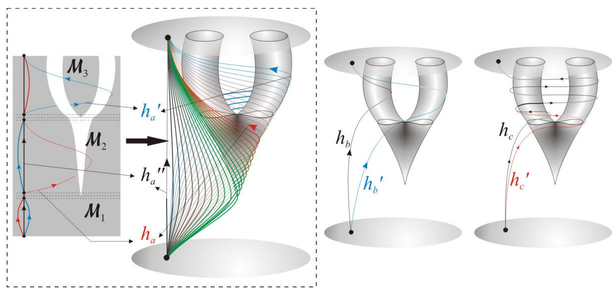

FIG. 3: The possible history-like paths of a particle in (a3)

From eq.(6), the history-like paths of N-identical particles in (c3) can be labeled by

$$\begin{aligned}[h]_{(\mathrm{a3})} &= \{W(\alpha,\beta,\sigma_i) \mid \text{if } W(\alpha,\beta,\sigma_i) \\ &= W''(\alpha,\beta,\sigma_i)([\alpha][\beta])^k W'(\sigma_i) \\ &\quad \text{then } W \sim W''(\alpha,\beta,\sigma_i)W'(\sigma_i); \\ &\quad \text{where } k = \text{integer}, \\ &\quad \text{and } i = 1,2,...,N-1\} \end{aligned} \quad (7)$$

Consider now other examples a bit different from the three previous ones investigated:

(a4) $\mathbb{R}^2$ changes into $\mathbb{R}^2$ with one puncture ($\mathbb{R}^2_\circ$), and then changes into $\mathbb{R}^2$ with two punctures ($\mathbb{R}^2_{\circ+\circ}$), but the second puncture does not emerge from the first one (not related with the previous puncture);

(a5) $\mathbb{R}^2$ changes into $\mathbb{R}^2_{\circ+\circ}$; two punctures are created simultaneously (not from a puncture that has been broken).

In (a4), there are no obstructions preventing history like paths to be deformed into trivial paths. Thus, the labels of its history-like paths are equal to the case in (a1) (eq.(3)).

A similar case also occurs in (a5). The topology change corresponds to two singular points on a slice M_c, so its history-like paths are also given by the relation (3).

Consider next M that experiences a spatial topology change once. Any slice M_c in M contains i singular points p_i ($i = 1, 2, ...$) that are non degenerate critical points on M_c. Consider a generator of a non-contractible homotopical loop $\mathcal{A}_{p_i} \in \mathcal{L}$. The p_i index in $\mathcal{A}_{p_i}$ express that this generator appears and/or disappears in the critical point p_i. The number of $\mathcal{A}_{p_i}$ on given p_i is labeled by $n(\mathcal{A})_{p_i}$.

Conjecture 1 *If there is only a $\mathcal{A}_{p_i}$ corresponding to a given p_i, $n(\mathcal{A})_{p_i} = 1$, then all of history-like paths $h_j(\mathcal{A}_{p_i}) \in \{h\}$ of an identical particle system traversing along a non-contractible homotopical loop generated by $\mathcal{A}_{p_i}$ can be homotopically deformed such that those paths will not pass over this non-contractible loop.*

Conjecture 2 *If there are $n(\mathcal{A})_{p_i} > 1$ corresponding to a given p_i then there is always any history-like path $h_j(\mathcal{A}_{p_i})$ of an identical particle system that can not be deformed homotopically into possible simpler path.*

Conjecture 3 *Consider $(2+1)$-dimensional topological space M. A history-like path $h_j(\mathcal{A}_{p_i}) \in \{h\}$ of a particle is labeled by $h_j(\mathcal{A}_{p_i}) = \eta_1(\mathcal{A}_{p_i}) \circ \eta_2(\mathcal{A}_{p_i}) \circ \eta_3(\mathcal{A}_{p_i}) \circ \cdots$. If associated with a $p_i \in M$, $n(\mathcal{A})_{p_i} > 1$ then a path $\eta_k(\mathcal{A}_{p_i})$, $k = 1, 2, ...$ will be an "obstruction" of a path $\eta_{k+l}(\mathcal{A}_{p_i})$, $l = 1, 2, ...$ such that $h_j(\mathcal{A}_{p_i})$ can not be deformed homotopically into a simpler path.*

Consider M with the following spatial topology changes:

(b1) S^2 changes into a disc D^2 ($S^2 \Rightarrow D^2$);

(b2) S^2 changes into a torus T^2 ($S^2 \Rightarrow T^2$).

For N-identical particles in (b1), corresponding to Conjecture 1, there is only one generator α related to a singular point p. Thus

$$[h]_{(\mathrm{b1})} = \{W(\sigma_i) \mid 1 \leqslant i \leqslant N-1\}. \quad (8)$$

For the case in (b2), The spatial topology change with a singular point emerges into two independent generators, α and β. Correspond to the Conjectures 2 and 3, and considering the relation (5), then

$$[h]_{(\mathrm{b2})} = \{W(\sigma_i, \alpha, \beta) \mid 1 \leqslant i \leqslant N-1\}. \quad (9)$$

IV. THE CASES WITH $(3+1)$-DIMENSIONAL M

Consider a topology change described as follow:

(c1) S^3 changes into $S^2 \times S^1$ ($S^3 \Rightarrow S^2 \times S^1$);

(c2) $\mathbb{R}^3$ changes into $\mathbb{R}^3$ with a 3-dimensional cylindrical hole having an infinite length; this cylindrical hole is then split into two cylindrical holes such that $\mathbb{R}^3$ finally possesses two cylindrical holes ($\mathbb{R}^3 \Rightarrow \mathbb{R}^3 - (\text{cyl-3D p}^\infty) \Rightarrow \mathbb{R}^3 - (\text{cyl-3D p}^\infty \sqcup \text{cyl-3D p}^\infty)$, where cyl-3D p^∞ is a 3-dimensional cylindrical hole with infinite length).

The (c1) can be divided into $\mathcal{M}_1 = S^3 \times I$ and $\mathcal{M}_2 = (S^2 \times S^1) \times I$. According to Conjecture 1, all of the history-like paths can be deformed homotopically into trivial paths. Thus, all of the history-like paths can be deformed into trivial paths,

$$[h]_{(\mathrm{c1})} = \{W(\sigma_i) \mid 1 \leqslant i \leqslant N-1\}. \quad (10)$$

The space-time (c2) can be divided into $\mathcal{M}_1 = \mathbb{R}^3 \times I$, $\mathcal{M}_2 = (\mathbb{R}^3 - (\text{sil-3D p}^\infty)) \times I$, and $\mathcal{M}_3 = (\mathbb{R}^3 - (\text{sil-3D p}^\infty \sqcup \text{sil-3D p}^\infty)) \times I$. The topology changes experienced are similar to the case in (a3). If the topology changes are considered from $\mathcal{M}_1$ to $\mathcal{M}_2$ and from $\mathcal{M}_2$ to $\mathcal{M}_3$ then there exist only one and two non-contractible homotopical loop, respectively, corresponding to a critical point on M_{c_1} and M_{c_2}, respectively. Therefore, corresponding to Conjecture 2, there always exist some history-like path that can not be deformed into a simpler or a trivial path. The paths are in a 4-dimensional topological space. Thus, any obstruction appearing as in a 2-dimensional space will disappear. So, the history-like paths in the (c2) can be labeled by

$$[h]_{(\mathrm{c2})} = \{W(\alpha, \sigma_i) \mid 1 \leqslant i \leqslant N-1\}. \quad (11)$$

V. CONCLUSION

For N identical particles in a topologically space-time undergoing spatial topology changes, the quantization of the system is constructed by introducing the "history-like homotopical path" concept constructed by the following reasons: First, in each slice of a topologically space-time, for all different times t, there exists a local region, which is homeomorphic to a local region in all other slices such that identification of points on these different regions can be done. Second, the topologically space-time can be divided into many regions whose not containing a singular slice and many regions containing singular slices. The history-like paths in the regions which are not containing a singular slice can be labelled as in the case when there is no topology change, while in the regions containing singular slices, any history-like path can be deformed so that it will occupy the regions whose not containing a singular slice. This can be done as long as the region containing singular slice is path connected. Thus, third, all of the history-like paths can be labelled by a composition of history-like paths of each region whose not containing a singular slice in topologically space-time. Equivalent classes of history-like paths in a region whose not containing a singular slice M_c, which are labelled by "words" W in the group theory meaning, are in a (1-1)-correspondence with the elements of $\pi_1(Q(\Sigma_i))$, where Si is a spatial space which is a deformation retract of $\mathcal{M}_i$.

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