On the Moments, Cumulants, and Characteristic Function of the Log-Logistic Distribution

Dian Ekawati[1], Warsono[1], and Dian Kurniasari[1]

Abstract—**This research examine about the moments, cumulants, and characteristic function of the log-logistic distribution. Therefore, the purposes of this article are (1) finding moments of the log-logistic distribution by using moment generating function and by definition of expected values of the log-logistic random variable and (2) finding the cumulants and characteristic function of the log-logistic distribution. Log-logistic distribution has two parameters: the shape parameter α and β as a parameter scale. Moments of the log-logistic distribution can be determined by using the moment generating function or the definition of expected value. Cumulants determined by the moments that have been found previously. Furthermore, skewness and kurtosis can be determined from the log-logistic distribution. While the characteristic function is the expected value of e^{itx}, which I as an imaginary number.**

Keywords—**Log-logistic Distribution, Moments, Cumulants, Characteristics Function.**

Abstrak— **Penelitian ini mengkaji tentang momen, kumulan, dan fungsi karakteristik dari distribusi log-logistik. Oleh karenanya tujuan dari tulisan ini adalah (1) menentukan moment ke-r distribusi log-logistik berdasarkan fungsi pembangkit momen, dan membuktikannya berdasarkan definisi nilai harapan dari distribusi log-logstik, dan (2) menentukan fungsi karakteristik distribusi log-logistik. Distribusi log-logistik memiliki dua parameter yaitu α sebagai parameter bentuk dan β sebagai parameter skala. Momen dari distribusi log-logistik dapat ditentukan dengan menggunakan fungsi pembangkit momen atau definisi dari nilai harapan. Kumulan ditentukan dengan momen yang telah ditemukan sebelumnya. Selanjutnya, dapat ditentukan skewness dan kurtosis dari distribusi log-logistik. Sedangkan fungsi karakteristik adalah nilai harapane^{itx}, dimana *I* sebagai angka imajiner.**

Kata Kunci—***Distribusi Log-logistik, Momen, Kumulan, FungsiKarakteristik.***

I. Introduction

A log-logistic distribution is a probability distribution of random variable that the logaritm has logistic distribution. Log-logistic distribution with two parameters, α as the form parameter and β as the scale parameter. According to [3] a random variable *X is* said to be a log-logistic distribution with the form parameter α and the scale parameter β, are denotated $X \sim L_L(\alpha,\beta)$, if Probability Density Function (PDF) is given by :

$$f(x;\alpha;\,\beta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2} \qquad (1)$$

where, $\alpha > 0$ and $\beta > 0$.

The log-logistic distribution has the same form with log-normal distribution but has heavier tail. Probability Density Function of log-logistic distribution In statistics, the log-logistic distribution is continuous probability distribution for non-negative random variable. For instances, death value caused by cancer diagnose or treatment, and also used in hydrology for rate of flow water model and rainfall, in economics as a simple model of wealth or incomedistribution.

Generally, the main focus in the research of the log-logistic distribution propertiesis in the areas of the expected value, variance, and quantile of this distribution. In this article we extend to derive properties of the log-logistic distribution in terms of the r-moment, cumulants, and characteristic function mathematically.

[1]Dian Ekawati[1], Warsono[1], and Dian Kurniasari are with Departement of Mathematics, Faculty of Mathematics and Natural Science, Universitas Lampung, Bandar Lampung, 35158, Indonesia. E-mail: diianekaa@gmail.com; chacho81@yahoo.com; dksari13@gmail.com.

Therefore, the purposes of this article are (1) Finding moments of the log-logistic distribution (α,β) by using Moment Generating Function (MGF) and proved by definition, that is through the expected value of the log-logistic random variable. (2) Finding the cumulants and characteristic function of the log-logistic distribution (α,β).

The second section of this article discuss about used methods. The Moment Generating Function (MGF) to find moment, cumulants, and characteristic function of the log-logistic distribution is determined in In the section 3.1. In the section 3.2 along with skewness function and kurtosis function, we generate the moments, cumulants, and characteristic function of the log-logistic distribution. In the section 3.3 graphically, we discuss the Probability Density Function (PDF), skewness and kurtosis function of the Log-logistic distribution. Finally, the last section is the conclusion of the article.

II. Method

This section discusses basic theories in mathematically deriving the moment, cumulants, and characteristic function of the log-logistic distribution (α,β).

A. *Moment Generating Function*

Moment generating function is used to determine r-moment of the X log-logistic random variable. Moment generating fnction is denoted $M_x(t)$.The moment generating function for the log-logistic distribution is given as follows [5]:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \qquad (1)$$

B. Moments

Average and variance actually are special kind of other measurements called moments. To generate these moments, we need to differentiate the moment generating function. The moments of the log-logistic distribution then can be retrieved as follows [9]:
The first moment,

$$M'_X(t) = \frac{d}{dt} M_X(t) \tag{3}$$

The second moment,

$$M''_X(t) = \frac{d^2}{dt^2} M_X(t) \tag{4}$$

The third moment,

$$M'''_X(t) = \frac{d^3}{dt^3} M_X(t) \tag{5}$$

The fourth moment,

$$M''''_X(t) = \frac{d^4}{dt^4} M_X(t) \tag{6}$$

The r-moment,

$$M^r{}_X(t) = \frac{d^r}{dt^r} M_X(t) \tag{7}$$

C. Expectation of the Random Variable

Let X be a continuous random variable having a probability density function f(x) such that we have certain absolute convergence[5]; namely,

$$\int_{-\infty}^{\infty} |x| f(x) dx \text{ exists} \tag{8}$$

The *expectation of a random variable* is

$$E(X) = \int_{-\infty}^{\infty} |x| f(x) dx \tag{9}$$

D. Cumulants

The other characteristics or properties of distributionscan be determined by their cumulantss. On calculating to determine the cumulants, we use moments that have been determined before[6]. The cumulantss are defined as follows:

$$K_1 = \mu'_1 \tag{10}$$

$$K_2 = \mu'_2 - {\mu'_1}^2 \tag{11}$$

$$K_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2{\mu'_1}^3 \tag{12}$$

$$K_4 = \mu'_4 - 4\mu'_1\mu'_3 - 3{\mu'_2}^2 + 12\mu'_2{\mu'_1}^2 - 6{\mu'_1}^4 \tag{13}$$

$$\vdots$$

$$K_r = \mu'_r - \sum_{n=1}^{r-1} \binom{r-1}{n-1} k_n \mu'_{r-n} \tag{14}$$

E. Characteristic Function

The characteristic function is one of important features in the probability and distribution concept. Similar to the moment generating function, the characteristic function could be used to calculate moments of the X random variable. The characteristic function can be defined as follows [8]:

$$E(e^{itX}) = \int_{-\infty}^{\infty} e^{itX}\ dPx = \int_{-\infty}^{\infty} e^{itX}\ dF(x) \tag{15}$$

where

$$e^{itX} = \cos tX + i \sin tX \tag{16}$$

III. Result and Discussion

A. Moment Generating Function of the Log-Logistic Distribution

The moment generating function of the log-logistic distribution should be determined before we determine the moments of the log-logistic distribution. The moment generating function for the log-logistic distribution is given as follow:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} e^{tx} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2} dx$$

By applying MacLaurin series [7] the equation becomes:

$$M_X(t) = \int_0^{\infty} \left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right) \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha} \left(\frac{x}{\beta}\right)^{-1} \left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2} dx$$

$$M_X(t) = \int_0^{\infty} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha}\left(\frac{x}{\beta}\right)^{-1}\left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2} dx + \int_0^{\infty} tx\, \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha}\left(\frac{x}{\beta}\right)^{-1}\left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2} dx + \int_0^{\infty} \frac{t^2x^2}{2!} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha}\left(\frac{x}{\beta}\right)^{-1}\left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2} dx + \int_0^{\infty} \frac{t^3x^3}{3!} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha}\left(\frac{x}{\beta}\right)^{-1}\left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2} dx +$$

by subtituting $y = \left(\frac{x}{\beta}\right)^{\alpha}$, we find $x = y^{\frac{1}{\alpha}}\beta$ with $dx = \frac{\beta}{\alpha} y^{\frac{1}{\alpha}-1} dy$.

It means that if boundary of x = 0 then boundary of y = 0 and for x = ∞ then y = ∞. Then, he moment generating function it can be written as follows:

$$M_X(t) = \int_0^{\infty} \frac{1}{(1+y)^2} dy + t\beta \int_0^{\infty} \frac{y^{\frac{1}{\alpha}}}{(1+y)^2} dy + t^2\beta^2 \int_0^{\infty} \frac{y^{\frac{2}{\alpha}}}{(1+y)^2} dy + t^3x^3 \int_0^{\infty} \frac{y^{\frac{3}{\alpha}}}{(1+y)^2} dy + \cdots$$

by using Beta function [1] :

$$M_X(t) = \boldsymbol{B}(1,1) + t\beta \boldsymbol{B}\left(\frac{\alpha+1}{\alpha}, \frac{\alpha-1}{\alpha}\right) + \frac{t^2\beta^2}{2!} \boldsymbol{B}\left(\frac{\alpha+2}{\alpha}, \frac{\alpha-2}{\alpha}\right) + \frac{t^3\beta^3}{3!} \boldsymbol{B}\left(\frac{\alpha+3}{\alpha}, \frac{\alpha-3}{\alpha}\right) + \cdots\cdots$$

Hence, the moment generating function of log-logistic distribution is:

$$M_X(t) = \sum_{n=0}^{\infty} \frac{(t\beta)^n}{n!} \boldsymbol{B}\left(\frac{\alpha+n}{\alpha}, \frac{\alpha-n}{\alpha}\right)$$

B. Moments, Cumulants, and Characteristic Function of Log-Logistic Distribution

1) Moments

By differentiating the moment generating function that we have before then moments of log-logistic distribution are retrieved as follow:
The first moment,

$$M'_X(t) = \frac{d}{dt} M_X(t)$$

$$M'_X(t) = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{(t\beta)^n}{n!} \boldsymbol{B}\left(\frac{\alpha+n}{\alpha}, \frac{\alpha-n}{\alpha}\right)$$

$$M'_X(t) = \sum_{n=0}^{\infty} \frac{nt^{n-1}\beta^n}{n!} \boldsymbol{B}\left(\frac{\alpha+n}{\alpha}, \frac{\alpha-n}{\alpha}\right)$$

$$M'_X(t=0) = \beta \boldsymbol{B}\left(\frac{\alpha+1}{\alpha}, \frac{\alpha-1}{\alpha}\right)$$

The second moment,

$$M''_X(t) = \frac{d^2}{dt^2} M_X(t)$$

$$M''_X(t) = \frac{d^2}{dt^2} \sum_{n=0}^{\infty} \frac{(t\beta)^n}{n!} \boldsymbol{B}\left(\frac{\alpha+n}{\alpha}, \frac{\alpha-n}{\alpha}\right)$$

$$M''_X(t) = \sum_{n=0}^{\infty} \frac{n\,(n-1)t^{n-2}\beta^n}{n!} \boldsymbol{B}\left(\frac{\alpha+n}{\alpha}, \frac{\alpha-n}{\alpha}\right)$$

$$M''_X(t=0) = \beta^2 \boldsymbol{B}\left(\frac{\alpha+2}{\alpha}, \frac{\alpha-2}{\alpha}\right)$$

The third moment,

$$M'''_X(t) = \frac{d^3}{dt^3} M_X(t)$$

$$M'''_X(t) = \frac{d^3}{dt^3}\sum_{n=0}^{\infty}\frac{(t\beta)^n}{n!}\boldsymbol{B}\left(\frac{\alpha+n}{\alpha},\frac{\alpha-n}{\alpha}\right)$$

$$M'''_X(t) = \sum_{n=0}^{\infty}\frac{n\,(n-1)(n-2)t^{n-3}\beta^n}{n!}\boldsymbol{B}\left(\frac{\alpha+n}{\alpha},\frac{\alpha-n}{\alpha}\right)$$

$$M'''_X(t=0) = \beta^3\boldsymbol{B}\left(\frac{\alpha+3}{\alpha},\frac{\alpha-3}{\alpha}\right)$$

Fourth Moment,

$$M''''_X(t) = \frac{d^4}{dt^4} M_X(t)$$

$$M''''_X(t) = \frac{d^4}{dt^4}\sum_{n=0}^{\infty}\frac{(t\beta)^n}{n!}\boldsymbol{B}\left(\frac{\alpha+n}{\alpha},\frac{\alpha-n}{\alpha}\right)$$

$$M''''_X(t) = \sum_{n=0}^{\infty}\frac{n\,(n-1)(n-2)(n-3)t^{n-4}\beta^n}{n!}\boldsymbol{B}\left(\frac{\alpha+n}{\alpha},\frac{\alpha-n}{\alpha}\right)$$

$$M''''_X(t=0) = \beta^4\boldsymbol{B}\left(\frac{\alpha+4}{\alpha},\frac{\alpha-4}{\alpha}\right)$$

The r-Moment,

$$M^r_X(t) = \frac{d^r}{dt^r} M_X(t)$$

$$M^r_X(t) = \frac{d^r}{dt^r}\sum_{n=0}^{\infty}\frac{(t\beta)^n}{n!}\boldsymbol{B}\left(\frac{\alpha+n}{\alpha},\frac{\alpha-n}{\alpha}\right)$$

$$M^r_X(t) = \sum_{n=0}^{\infty}\frac{n\,(n-1)(n-2)\cdots(n-(r-1))(n-r)t^{n-r}\beta^n}{n!}\boldsymbol{B}\left(\frac{\alpha+n}{\alpha},\frac{\alpha-n}{\alpha}\right)$$

$$M^r_X(t=0) = \beta^r\boldsymbol{B}\left(\frac{\alpha+r}{\alpha},\frac{\alpha-r}{\alpha}\right)$$

Then, the above equations of momentsmight also be proven by the following definition of the expected values:

The first moment,

$$E(X) = \int_0^\infty x\, f(x)dx$$

$$= \int_0^\infty x\,\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2}dx$$

$$= \int_0^\infty x\,\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha}\left(\frac{x}{\beta}\right)^{-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2}dx$$

Let $y = \left(\frac{x}{\beta}\right)^{\alpha}$ then, $x = y^{\frac{1}{\alpha}}\beta$. Hence $dx = \frac{\beta}{\alpha}y^{\frac{1}{\alpha}-1}dy$.

It means that if boundary of x = 0 then boundary of y = 0 andfor x = ∞then y = ∞. We have the result as

$$E(X) = \beta\int_0^\infty \frac{y^{\frac{1}{\alpha}}}{(1+y)^2}\,dy$$

By using Beta functions, the first expectation of the log-logistic is:

$$E(X) = \beta\,\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)$$

The second expectation,

$$E(X^2) = \int_0^\infty x^2 f(x)dx$$

By similar processes in the first expectation then we have the second expectation of log-logistic distribution as follows:

$$E(X^2) = \beta^2\,\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right)$$

The third expectation,

$$E(X^3) = \int_0^\infty x^3 f(x)dx$$

By similar processes in the second expectation then we have the third expectation of log-logistic distribution as follows:

$$E(X^3) = \beta^3\,\boldsymbol{B}\left(\frac{\alpha+3}{\alpha},\frac{\alpha-3}{\alpha}\right)$$

The fourth expectation,

$$E(X^4) = \int_0^\infty x^4 f(x)dx$$

By similar processes in the third expectation then we have the fourth expectation of log-logistic distribution as follows:

$$E(X^4) = \beta^4\,\boldsymbol{B}\left(\frac{\alpha+4}{\alpha},\frac{\alpha-4}{\alpha}\right)$$

The r-expectation,

$$E(X^r) = \int_0^\infty x^r f(x)dx$$

Finally, by above processesin fourth expectation then we find r-expectation of the log-logistic distribution as follows:

$$E(X^r) = \beta^r\,\boldsymbol{B}\left(\frac{\alpha+r}{\alpha},\frac{\alpha-r}{\alpha}\right)$$

2) Cumulantsof the Log-logistic Distribution

The following equations are the first, second, third, fourth, and r- cumulantsofthe log-logistic distribution:

$$K_1 = \mu'_1 = \beta\,\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)$$

$$K_2 = \mu'_2 - {\mu'_1}^2 = \beta^2\left(\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right) - \left(\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\right)^2\right)$$

$$K_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2{\mu'_1}^3 = \beta^3\begin{pmatrix}\boldsymbol{B}\left(\frac{\alpha+3}{\alpha},\frac{\alpha-3}{\alpha}\right)\\ -3\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right)\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\\ +2\left(\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\right)^3\end{pmatrix}$$

$$K_4 = \mu'_4 - 4\mu'_1\mu'_3 - 3{\mu'_2}^2 + 12\mu'_2{\mu'_1}^2 - 6{\mu'_1}^4$$

$$= \beta^4\left(\boldsymbol{B}\left(\frac{\alpha+4}{\alpha},\frac{\alpha-4}{\alpha}\right) - 4\,\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\boldsymbol{B}\left(\frac{\alpha+3}{\alpha},\frac{\alpha-3}{\alpha}\right) - 3\left(\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right)\right)^2 + 12\,\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right)\left(\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\right)^2 - 6\left(\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\right)^4\right)$$

$$K_r = \mu'_r - \sum_{n=1}^{r-1}\binom{r-1}{n-1}k_n\mu'_{r-n}$$

$$= \beta^r\,\boldsymbol{B}\left(\frac{\alpha+r}{\alpha},\frac{\alpha-r}{\alpha}\right) - \sum_{n=1}^{r-1}\binom{r-1}{n-1}K_n\,\beta^{r-n}\,\boldsymbol{B}\left(\frac{\alpha+(r-n)}{\alpha},\frac{\alpha-(r-n)}{\alpha}\right)$$

Thus, skewness and kurtosis of the log-logistic distribution can be determined as follows

$$Skew[X] = \frac{E[(X-\mu)^3]}{E[(X-\mu)^2]^{\frac{3}{2}}} = \frac{K_3}{(K_2)^{\frac{3}{2}}}$$

$$Skew[X] = \frac{\begin{pmatrix}\boldsymbol{B}\left(\frac{\alpha+3}{\alpha},\frac{\alpha-3}{\alpha}\right)\\ -3\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right)\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\\ +2\left(\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\right)^3\end{pmatrix}}{\left(\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right) - \left(\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\right)^2\right)^{\frac{3}{2}}}$$

Kurtosis,

$$\alpha_4 = \frac{\mu'_4}{\sigma^4}$$

$$\alpha_4 = \frac{\begin{pmatrix}\boldsymbol{B}\left(\frac{\alpha+4}{\alpha},\frac{\alpha-4}{\alpha}\right)\\ -4\,\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\boldsymbol{B}\left(\frac{\alpha+3}{\alpha},\frac{\alpha-3}{\alpha}\right)\\ -3\left(\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right)\right)^2\\ +12\,\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right)\left(\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\right)^2\\ -6\left(\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\right)^4\end{pmatrix}}{\left(\left(\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right) - \left(\beta\,\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right)\right)^2\right)\right)^2}$$

3) Characteristic Function of the Log-Logistic Distribution

The characteristic function of the log-logistic distribution can be expressed as follows:

$$\Phi_X(t) = \int_0^\infty e^{itx} f(x)dx$$

$$= \int_0^\infty \cos tx + i \sin tx \left\{\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2}\right\} dx$$

Equation above made of 2 part to solve it one by one:
Part I,

$$\int_0^\infty \cos\ tx \left\{\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2}\right\} dx$$

Part II,

$$\int_0^\infty i \sin tx \left\{\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2}\right\} dx$$

Furthermore, by transforming sin *tx* and cos *tx* using MacLaurin series, then we may find results of each part as follows:

$$\text{I} = \boldsymbol{B}(1,1) - \frac{t^2\beta^2}{2!}\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right) + \frac{t^4\beta^4}{4!}\boldsymbol{B}\left(\frac{\alpha+4}{\alpha},\frac{\alpha-4}{\alpha}\right) + \frac{t^6\beta^6}{6!}\boldsymbol{B}\left(\frac{\alpha+6}{\alpha},\frac{\alpha-6}{\alpha}\right) + \cdots$$

$$\text{II} = it\beta\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{a}\right) - \frac{it^3\beta^3}{3!}\boldsymbol{B}\left(\frac{\alpha+3}{\alpha},\frac{\alpha-3}{a}\right) + \frac{it^5\beta^5}{5!}\boldsymbol{B}\left(\frac{\alpha+5}{\alpha},\frac{\alpha-5}{a}\right) - \frac{it^7\beta^7}{7!}\boldsymbol{B}\left(\frac{\alpha+7}{\alpha},\frac{\alpha-7}{a}\right) + \cdots$$

$$\Phi(x) = \text{I} + \text{II}$$

$$\Phi_X(t) = \boldsymbol{B}(1,1) + it\beta\boldsymbol{B}\left(\frac{\alpha+1}{\alpha},\frac{\alpha-1}{\alpha}\right) - \frac{t^2\beta^2}{2!}\boldsymbol{B}\left(\frac{\alpha+2}{\alpha},\frac{\alpha-2}{\alpha}\right) - \frac{it^3\beta^3}{3!}\boldsymbol{B}\left(\frac{\alpha+3}{\alpha},\frac{\alpha-3}{\alpha}\right) + \frac{t^4\beta^4}{4!}\boldsymbol{B}\left(\frac{\alpha+4}{\alpha},\frac{\alpha-4}{\alpha}\right) + \frac{it^5\beta^5}{5!}\boldsymbol{B}\left(\frac{\alpha+5}{\alpha},\frac{\alpha-5}{\alpha}\right) - \cdots$$

The characteristic function of the log-logistic distribution is :

$$\Phi_X(t) = \sum_{n=0}^{\infty}\frac{(it\beta)^n}{n!}\boldsymbol{B}\left(\frac{\alpha+n}{\alpha},\frac{\alpha-n}{\alpha}\right)$$

We also may find the norm characteristic function log-logistic distribution:

$$|\Phi_X(t)| = \boldsymbol{B}(1,1) = 1$$

We note that the value of norm characteristic function is equal to 1. It means that characteristic function of the log-logistic distribution is a finite function. This result can be seen from graphs as discussed in the following sections.

C. Graphs of the Probability Density Function, Skewness Function, and Kurtosis Function

1) Probability Density Function of the Log-Logistic Distribution

Based on Figure 1, graphs of the probability density function of the log-logistic distribution for α = 35 and β = 5 (red curve), for α = 45 and β = 3 (blue curve), for α = 55 and β = 1 (purple curve), it is seen that when β getting decline the concentration data is getting move to the left, but when α getting rise the form curve of the log-logistic distribution is getting pointed.

2) Skewness Function

Based on Figure 2, graphs of skewness functionof the log-logistic distribution that we found by entering 4 different values of α, then the curve formed on x and y with positive values. The skewness curve of the log-logistic distribution is formed on y positive values. It means that the skewnessof the log-logistic distribution is positive (*skewed to the right*).

3) Kurtosis Function

Graphs of kurtosis function of the log-logistic distribution are created by entering 4 different values of α. Based on Figure 3, it can be seen that for values of kurtosis less than 3 the distribution tends to see as platy curtic distribution, and for values of kurtosis greater than 3 is leptokurtic distribution.

III. Conclusion

Based on above results, it can be concluded that

1. The r-moment of the log-logistic distribution (α,β)

$$M^r{}_X(t|0) = \beta^r\boldsymbol{B}\left(\frac{\alpha+r}{\alpha},\frac{\alpha-r}{\alpha}\right)$$

2. The r-cumulantof thelog-logistic distribution (α,β) is

$$K_r = \mu_r' - \sum_{n=1}^{r-1}\binom{r-1}{n-1}k_n\mu_{r-n}'$$

$$= \beta^r\boldsymbol{B}\left(\frac{\alpha+r}{\alpha},\frac{\alpha-r}{\alpha}\right) - \sum_{n=1}^{r-1}\binom{r-1}{n-1}K_n\beta^{r-n}\boldsymbol{B}\left(\frac{\alpha+(r-n)}{\alpha},\frac{\alpha-(r-n)}{\alpha}\right)$$

3. The characteristic function of the log-logistic distribution (α,β) is

$$\Phi_X(t) = \sum_{n=0}^{\infty}\frac{(it\beta)^n}{n!}\boldsymbol{B}\left(\frac{\alpha+n}{\alpha},\frac{\alpha-n}{\alpha}\right)$$

Acknowledgement

The authors would like to thank anonymous referees andeditors for valuable comments on the earlier version of thisarticle, and that have significantly improved this article.

References

[1] M Abramowits and I A Stegun, *Handbook of Mathematical Functionswith Formulas, Graps, and Mathematical Tables*, 9th ed. Washington D.C, USA: U.S Goverment Printing, 1972.

[2] Asha Dixit, "Exact Comparison of Hazard Rate Functions of Log-Logistic Survival Distributions," Auburn University, Auburn, Alabama, Master Thesis 2008.

[3] M C M De Gunst, "Statistische Data Analyse," Vrije Universiteit, Amsterdam, 1994.

[4] Edward J Dudewicz and Satya N Mishra, *StatistikaMatematika Modern*. Bandung, Indonesia: ITB Bandung, 1995.

[5] Robert V Hogg and Allen T Craig, *Introduction to Mathematical Statistic*, 5th ed. New Jersey, USA: Prentice Hall, 1995.

[6] G Maurice Kendall and A Stuart, *The Advanced Theory of Statistics (Vol I)*. New York, USA: MacMillan, 1977.

[7] Louis Leithold, *The Calculus, with Analytic Geometry: Functions of One Variable and Plane Analytic Geometry.* . New York, USA: Harper and Row, 1968.

[8] E Lukacs and R G Laha, *Applications of Characteristic Functions*. New York, USA: Hafner Publishing Company, 1964.

[9] Irwin Miller and Marylees Miller , *John E. Freund's Mathematical Statistics, Sixth Edition*. New Jersey, USA: Prentice Hall, 1998.

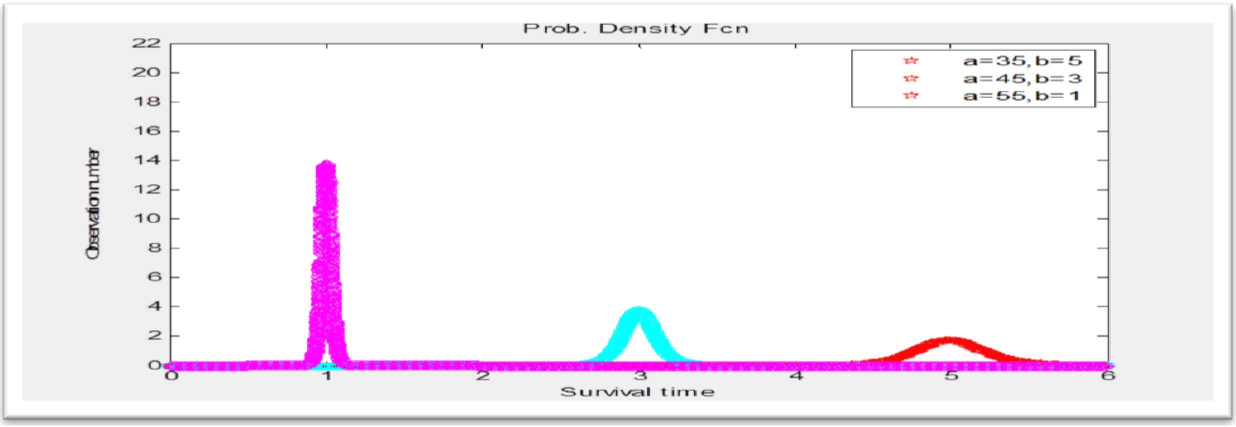

Figure 1. Graphs ofProbability Density Functionof Log-logistic Distributionwith α Riseand β Decline.

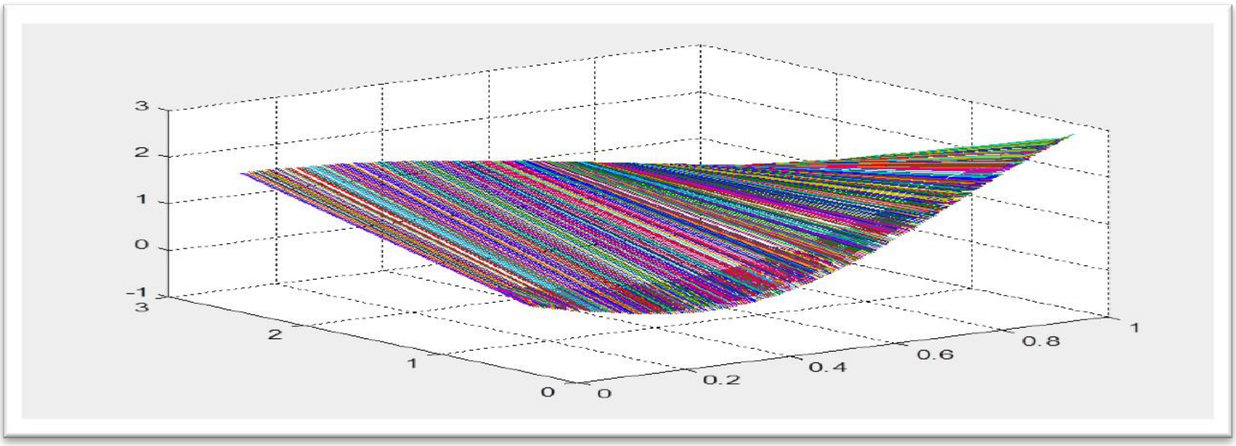

Figure 2.Graphs of Skewness Function of Log-logistic Distribution withMatrixof 4 values of α

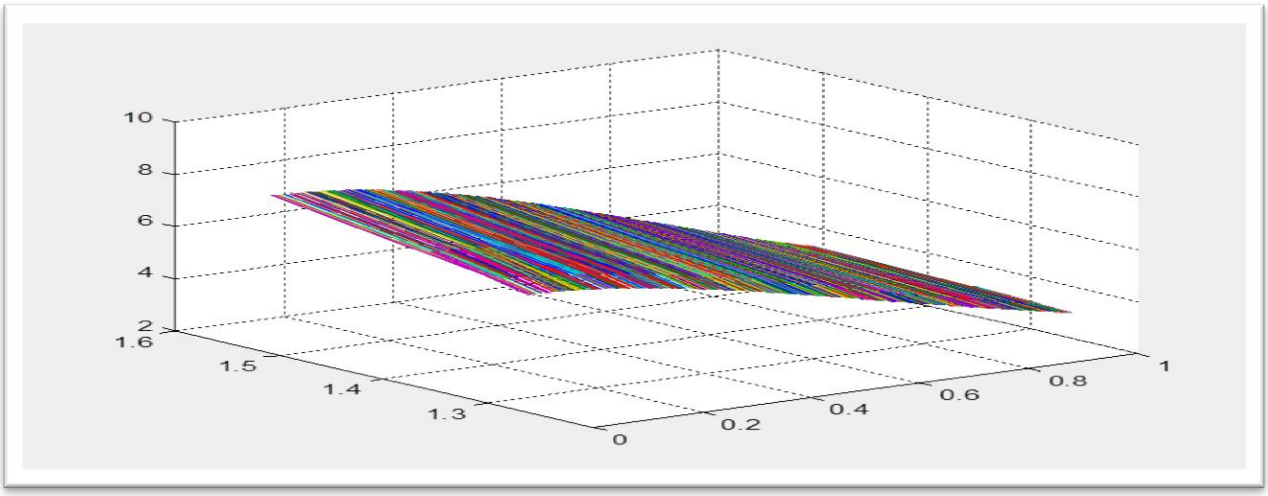

Figure 3.Graphs of Kurtosis Function of Log-logistic Distribution withMatrixof 4 values of α.