Eccentric Digraph
of Cocktail Party Graph and Hypercube

Tri Atmojo Kusmayadi and Nugroho Arif Sudibyo[1]

Abstract—**Let G be a graph with a set of vertices V(G) and a set of edges E(G). The distance from vertex u to vertex v in G, denoted by d(u, v), is the length of the shortest path from vertex u to v. The eccentricity of vertex u in graph G is the maximum distance from vertex u to any other vertices in G, denoted by e(u). Vertex v is an eccentric vertex from u if d(u, v) = e(u). The eccentric digraph ED(G) of a graph G is a graph that has the same set of vertices as G, and there is an arc (directed edge) joining vertex u to v if v is an eccentric vertex from u. In this paper, we determine the eccentric digraph of a class of graph called the cocktail party graph and hypercube.**

Keywords—**cocktail party graph, eccentric digraph, eccentricity, hypercube**

Abstrak—*Misal G adalah suatu graf dengan himpunan vertex V(G) dan himpunan vertex E(G) . Jarak dari vertex u ke vertex v di G, dinotasikan d(u,v), adalah panjang dari path terpendek dari vertex u ke v. Eksentrisitas vertex u dalam graf G adalah jarak maksimum dari vertex u ke sebarang vertex yang lain di G, dinotasikan e(u). Vertex v adalah suatu vertex eksentrik dari u jika d(u,v)=e(u). Digraf eksentrik ED(G) dari suatu graf G adalah suatu graf yang mempunyai himpunan vertex yang sama dengan himpunan vertex G, dan terdapat suatu arc (edge berarah) yang menghubungkan vertex u ke v jika v adalah suatu vertex eksentrik dari u. Pada makalah ini, akan diselidiki digraph eksentrik dari suatu kelas graf yaitu graf cocktail party dan graf hypercube.*

Kata Kunci—*graf cocktail party, digraf eksentrik, eksentrisitas, hipercube*

I. INTRODUCTION

Most of the notations and terminologies follow that of [1-2]. Let *G* be a graph with a set of vertices V(G) and a set of edges E(G). The distance from vertex *u* to vertex *v* in G, denoted by $d(u,v)$, is the length of the shortest path from vertex *u* to *v*. If there is no a path joining vertex u and vertex v, then $d(u,v) = \infty$. The eccentricity of vertex *u* in graph G is the maximum distance from vertex *u* to any other vertices in G, denoted by $e(u)$, and so $e(u) = \max\{d(u, v)| v \in V(G)\}$. Radius of a graph G, denoted by $rad(G)$, is the minimum eccentricity of every vertex in G. The diameter of a graph G, denoted by $diam(G)$, is the maximum eccentricity of every vertex in G. If $e(u) = rad(G)$, then vertex u is called central vertex. Center of a graph G, denoted by $cen(G)$, is an induced subgraph formed from central vertices of G. Vertex *v* is an eccentric vertex from *u* if $d(u,v)=e(u)$. The eccentric digraph ED(G) of a graph G is a graph that has the same set of vertices as G, V(ED(G)) = V(G), and there is an arc (directed edge) joining vertex *u* to *v* if *v* is an eccentric vertex from *u*. An arc of a digraph D joining vertex *u* to *v* and vertex *v* to *u* is called a symmetric arc. Further, Fred Buckley in [3] concluded that almost in every graph G, its eccentric digraph is ED(G) = $\bar{G}^*$, where $\bar{G}^*$ is a complement of G which is every edge replaced by a symmetric arc.

One of the topics in graph theory is to determine the eccentric digraph of a given graph. The eccentric digraph of a graph was initially introduced by Fred Buckley [3]. Some authors have investigated the problem of finding the eccentric digraph. For example, [3] determined the eccentric digraph of a digraph, while [4] found the characterisation of the eccentric digraphs. [3] also proposed an open problem to find the eccentric digraph of various classes of graphs. Some results related to this open problem can be found in [5-8].

In this paper, we also answer the open problem proposed by [3]. In particular, we determine the eccentric digraph of a graph called the cocktail party graph and hypercube.

II. METHOD

The materials of this research are mostly from the papers related to the eccentric digraph. There are three steps to determine the eccentric digraph from the given graph. The first step, we determined the distance from vertex *u* to any vertex *v* in the graph, denoted by $d(u, v)$, using Breadth First Search (BFS) Moore Algorithm taken from [1] as follows:

a. Take any vertex, say *u*, and labeled 0 which states the distance from *u* to it self, and other vertices are labeled ∞

b. All vertices having label ∞ adjacent to *u* are labeled by 1

c. All vertices having label ∞ adjacent to 1 are labeled by 2 and so on until the required vertex, say *v*, has already labeled.

The second step, we determined the vertex eccentricity *u* by choosing the maximum distance from the vertex *u*, and so we obtained the eccentric vertex *v* from *u* if $d(u, v) = e(u)$.

The final step, by joining an arc from vertex *u* to its eccentric vertex, so we obtained the eccentric digraph from the given graph.

III. RESULTS AND DISCUSSION

A. Eccentric Diagraph of Cocktail Party Graph

The cocktail party graph $H_{m,n}$, $m,n \geq 2$, as the graph with a vertex set $V=\{v_1,v_2,...,v_{mn}\}$ partitioned into *n* independent sets $V=\{I_1,I_2,...,v_n\}$ each of size *m* such that $v_iv_j \in E$ for all $i, j \in \{1,2,...,2n\}$ where $i \in I_p, j \in I_q, p \neq q$.

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The cocktail party graphs $H_{2,3}$, $H_{3,3}$, and $H_{3,4}$, can be described in Figure 1, Figure 2, and Figure 3, respectively [9].

Lemma 1.

Let $H_{m,n}$ be a cocktail party graph for $m,n \geq 2$, then the eccentricity of vertex $u_{i,j}$, $e(u_{i,j})= 2$, for $i=1,2...,m$ and $j=1,2,...,n$.

Proof.

From Table 1, it is easy to see that the farthest distance from vertex u_{ij} to vertex u_{ik} is 2, for $j \neq k$, so the eccentricity of u_{ij}, $e(u_{i,j})$, is 2.

Lemma 2.

Let $H_{m,n}$ be a cocktail party graph for $m,n \geq 2$, then the eccentric vertex of vertex $u_{i,j}$ is $u_{i,k}$, for $i=1,2...,m$ and $j=1,2,...,n$.

Proof.

By Lemma 1, the eccentricity of u_{ij} is 2, so the eccentric vertex $u_{i,j}$ is vertex u_{ik}, for $j \neq k$, $i=1,2...,m$, $j,k =1,2,...,n$.

Lemma 3.

Let $H_{m,n}$ be a cocktail party graph for $m,n \geq 2$, then its eccentric digraph has the vertex set

$$V(H_{m,n}) = \{u_{11}, u_{12}, \ldots, u_{nm}\}$$

and the arc set

$$A(ED(H_{m,n})) = \{\overleftrightarrow{u_{ij}u_{ik}} / i = 1,2,\ldots,m, \ j = 1,2,\ldots,n,$$

$$j \neq k, k = 1,2,\ldots,n\}.$$

where arcs $\overleftrightarrow{u_{ij}u_{ik}}$, $i = 1,2,\ldots,m$, $j=1,2,\ldots,n$, $k=1,2,...,n$, $j \neq k$, are symmetric arcs.

Proof.

The arcs can be obtained by joining every vertex to its eccentric vertex of a cocktail party graph $H_{m,n}$. By Lemma 2, the eccentric vertex of vertex u_{ij} is vertex u_{ik} for $j \neq k$, so u_{ij} is joined to u_{ik} forming a symmetric arc $\overleftrightarrow{u_{ij}u_{ik}}$. Hence, the eccentric digraph of $H_{m,n}$ has the vertex set $V(ED(H_{m,n}))=V(H_{m,n})$ and the arc set $A(ED(H_{m,n}))$.

Theorem 4.

Let $H_{m,n}$ be a cocktail party graph for $m,n \geq 2$, then the eccentric digraph of $H_{m,n}$,

$ED(H_{m,n})$ is a digraph nK_m having the vertex set

$$V(H_{m,n}) = \{u_{11}, u_{12}, \ldots, u_{nm}\}$$

and the arc set

$$A(ED(H_{m,n})) = \{\overleftrightarrow{u_{ij}u_{ik}} / i = 1,2,\ldots,m, \ j = 1,2,\ldots,n,$$

$$j \neq k, k = 1,2,\ldots,n\}.$$

Proof.

By Lemma 3, the arc from vertex u_{ij} to vertex u_{ik}, for $j \neq k$, are symmetric. So, the obtained digraph is a digraph nK_m having the vertex set $V(ED(H_{m,n}))$ and the arc set $A(ED(H_{m,n}))$. This completes the proof of the theorem. The eccentric digraphs of cocktail party graphs $H_{2,3}$, $H_{3,3}$, and $H_{3,4}$ are shown in the Figure 4, 5, and 6, respectively.

B. Eccentric Diagraph of Hypercupe

The hypercube Q_n is the graph whose vertex set is $\{0,1\}^n$ and where two vertices are adjacent if they differ in exactly one coordinate. Let $V(Q_n) = \{v_1, v_2,...., v_{2^n}\}$ and let $v_i \in V(Q_n)$ with $v_i = (v_{i1}, v_{i2}, ..., v_{in})$, $v_{ij} = 0$ or 1. Then $\overline{v}_i = (\overline{v}_{i1}, \overline{v}_{i2}, \ldots, \overline{v}_{in})$ is a vertex of $V(Q_n)$ with $\overline{v}_{ij} \neq v_{ij}$, $i, j = 0$ or 1 [2]. The hypercube Q_1, Q_2, and Q_3 can be described in Figure 7, Figure 8 and Figure 9, respectively.

Theorem 5. *Let Q_n be a hypercube for $n \geq 2$, then the eccentric digraph of Q_n, $ED(Q_n)$ is a digraph $2^{n-1}K_2$ having vertex set $V(Q_n) = \{v_1, v_2,...., v_{2^n}\}$ and the arc set $A(ED(Q_n)) = \overleftrightarrow{v_i \overline{v}_i}$, $i = 1,2,\ldots,2^{n-1}$ where,*

arc $\overrightarrow{v_i \overline{v}_i}$, $i = 1,2,\ldots,2^{n-1}$, are symmetric arcs

Proof. From Table 2, it is easy to see that the farthest distance between any two vertices of hypercube Q_n is n. In addition, the arcs are from vertex v_i to vertex $\overline{v_i}$ for every $i = 1,2, \ldots, 2^{n-1}$, and all of the arcs are symmetric. Based on the observation, it can be obtained that the eccentric digraph of $ED(Q_1)$ is a digraph K_2, $ED(Q_2)$ is a digraph $2K_2$, $ED(Q_3)$ is a digraph $4K_2$, and $ED(Q_4)$ is a digraph $8K_2$. Hence, the eccentric digraph of Q_n is a digraph $2^{n-1} K_2$. So, the obtained digraph is a digraph having the vertex set $V(Q_n)$ and the arc set $A(ED(Q_n))$. This completes the proof of the theorem. The eccentric digraphs of cocktail party graphs Q_1, Q_2 and Q_3 are shown in the Figure 10, Figure 11, and Figure 12, respectively.

IV. CONCLUSION

The results show that the eccentric digraph $ED(H_{m,n})$ of a cocktail party graph $H_{m,n}$ for $m,n \geq 2$, is a digraph nK_m having the vertex set $V(H_{mn}) = \{u_{11}, u_{12},....., u_{nm}\}$ and the arc set $A(ED(H_{m,n})) = \{u_{ij}u_{ik} / i\text{-}1,2,\ldots, m \, j - 1,2,\ldots, n \, j \neq k, k = 1,2, \ldots,n\}$ and the eccentric digraph $ED(Q_n)$ of a hypercube graph Q_n is a digraph $2^{n-1} K_2$ having the vertex set $V(Q_n) = \{v_1, v_2, \ldots, v_{2^n}\}$ and the arc set $A(ED(Q_n)) = v_i v_i, i = 1,2, \ldots, 2^{n-1}$.

As mentioned in previous sections the main goal of this paper is to find the eccentric digraph of a given class of graph. Some authors have conducted research on this problem. Most of them have left some open problems on their paper for the future research. We suggest readers to investigate the problem proposed by [3] by considering other classes of graphs.
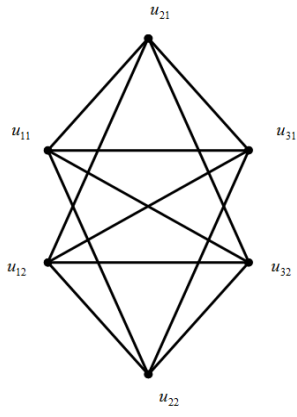
Figure 1. The cocktail party graph $H_{2,3}$
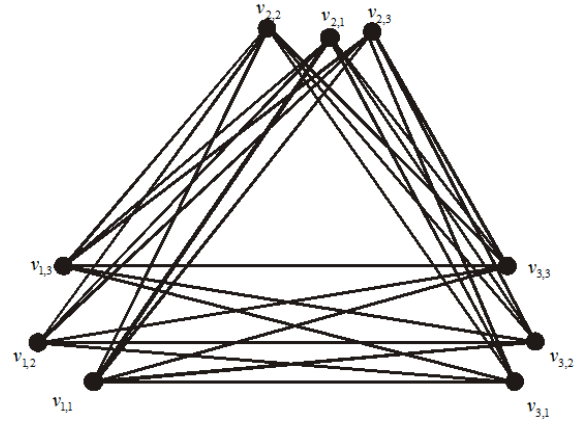
Figure 2. The cocktail party graph $H_{3,3}$

Figure 3. The cocktail party graph $H_{3,4}$

$$u_1 \quad u_2 \quad u_3$$

$$v_1 \quad v_2 \quad v_3$$

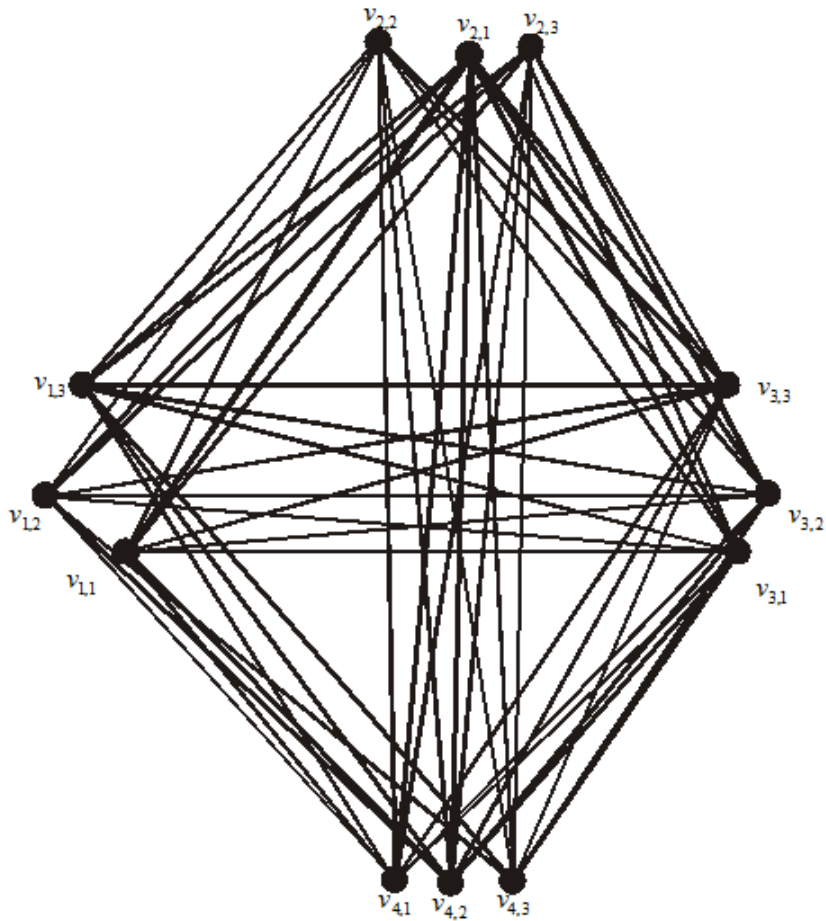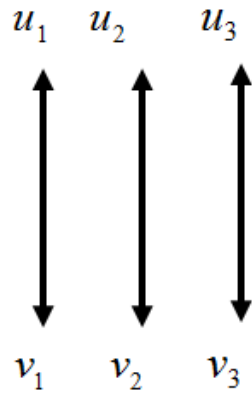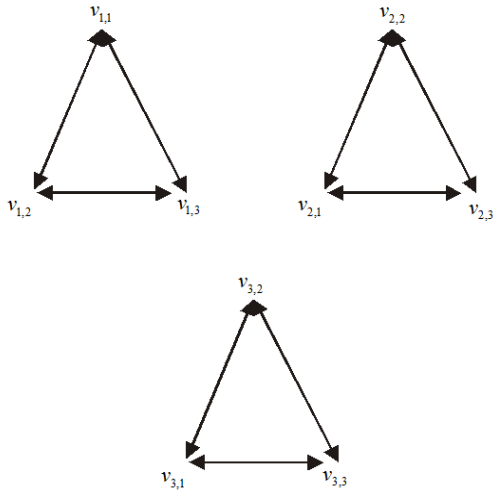Figure 4. The eccentric digraph of cocktail party graph $H_{2,3}$

Figure 5. The eccentric digraph of cocktail party graph $H_{3,3}$

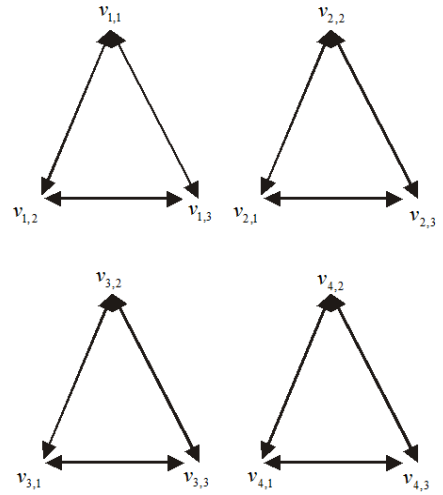Figure 6. The eccentric digraph of cocktail party graph $H_{3,4}$

$$v_1$$
$$0$$

$$1$$
$$v_2 = \bar{v_1}$$

Figure 7. The hypercube graph Q_1

v_1
00

v_2
01

10
$v_3 = \overline{v_2}$

11
$v_4 = \overline{v_1}$
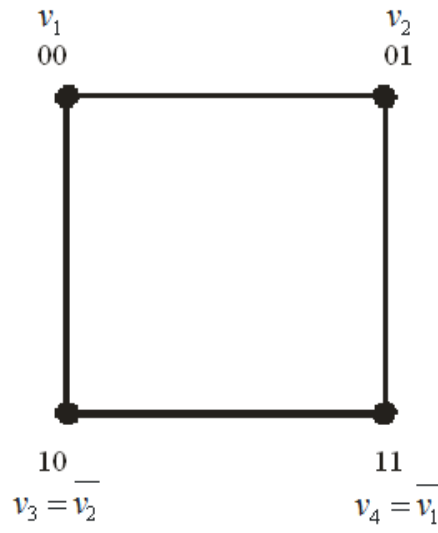
Figure 8. The hypercube graph Q_2

$v_5 = \overline{v_4}$
100

$v_6 = \overline{v_3}$
101

v_1
000

v_2
001

010
v_3

011
v_4

110
$v_7 = v_2$

111
$v_8 = v_1$
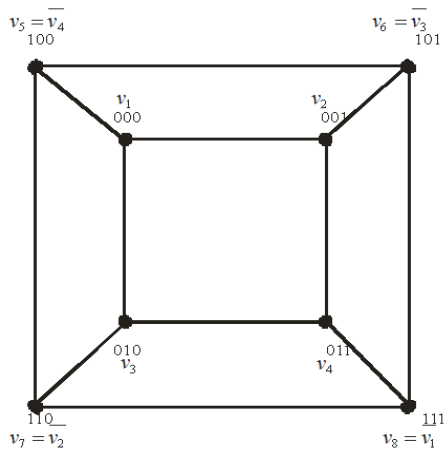
Figure 9. The hypercube graph Q_3

v_1
0

1
$v_2 = \overline{v_1}$

Figure 10. The eccentric digraph of hypercube graph Q_1

v_1
00

v_2
01

10
$v_3 = \overline{v_2}$

11
$v_4 = \overline{v_1}$
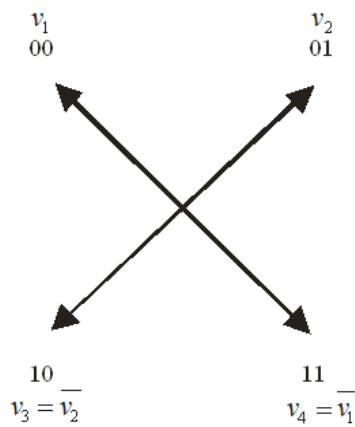
Figure 11. The eccentric digraph of hypercube graph

$$v_1 \qquad\qquad v_8 = \overline{v_1}$$
$$000 \longleftrightarrow 111$$

$$v_2 \qquad\qquad v_7 = \overline{v_2}$$
$$001 \longleftrightarrow 110$$

$$v_3 \qquad\qquad v_6 = \overline{v_3}$$
$$010 \longleftrightarrow 101$$

$$v_4 \qquad\qquad v_5 = \overline{v_4}$$
$$011 \longleftrightarrow 101$$
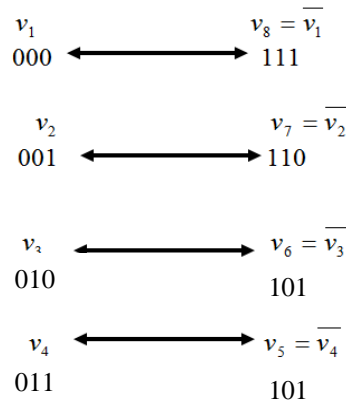
Figure 12. The eccentric digraph of hypercube graph Q_3

TABLE 1.
DISTANCE BETWEEN ANY TWO VESICLE OF COCKTAIL PARTY GRAPH $H_{M,N}$

vertex	$v_{1,1}$,	…	$v_{1,n}$	$v_{2,1}$,	…	$v_{2,n}$	…	$v_{m,1}$,	…	$v_{m,n}$
$v_{1,1}$	0	…	2	1	…	1	…	1	…	1
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
$v_{1,n}$	2	…	0	1	…	1	…	1	…	1
$v_{2,1}$	1	…	1	0	…	2	…	1	…	1
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
$v_{2,n}$	1	…	1	2	…	0	…	1	…	1
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
$v_{m,1}$	1	…	1	1	…	1	…	0	…	2
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
$v_{m,n}$	1	…	1	1	…	1	…	2	…	0

TABLE 2.
DISTANCE BETWEEN ANY TWO VESICLE OF GRAPH QN

vertex	v_1	v_2	...	v_2^n
v_1	0	1	...	*n-1*
v_2	1	0	...	…
.	.	.	...	
.	.	.	...	…
.	.	.	...	
v_2^n	*n*	*n-1*	...	0

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