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Minimum-Energy Control of Two-Link Manipulator with Pure State Constraints

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Abstract

This paper presents an analysis and numerical solutions of the minimum-energy control of two-link robot manipulator. The minimum-energy control point-to-point trajectory is investigated subject to control constraints and state constraints on the angular velocities. The numerical solutions are solved by transforming the original problem into a nonlinear programming problem. The mathematical analysis of the optimal control problems is done based on the numerical results using an indirect method. The necessary conditions can be stated as a multi-point boundary value problems.

Keywords*: optimal control, minimum-energy control, direct method, indirect method, robot manipulator*

1. Introduction

Robot manipulators are used for variety tasks in industry. The important performances of the robot manipulators are the speed and energy when its work. Therefore minimum energy point-to-point trajectory of two-link robot manipulators are investigated subject to

control and state constraints. The point-to-point control of multiple link manipulators can be applied to accurate aiming of an industrial robot or a multi-body spacecraft.

The methods for solving optimal control can be classified generally into two main categories: direct and indirect methods. The direct method solve the optimal control problem by discretizing the control and/or the state variables, transforming the optimal control problem into a Nonlinear Programming Problem, NLP, see eg Betts [1], Von Stryk and Bulirsch [2], Seywald and Kumar [3], while the indirect methods are based on the solving of the necessary conditions derived from the Pontryagin Maximum Principles (Pontryagin et al. [4]).

2. Optimal Control Problem

The general optimal control problem is to find an admissible control $\boldsymbol{u}$ to optimise the performance index in the following general form (Bryson and Ho [5]):

$$\mathcal{J}[\boldsymbol{u}] = \phi[\boldsymbol{x}(t_f, t_f)] + \int_{t_0}^{t_f} L(\boldsymbol{x}, \boldsymbol{u}, t)dt \tag{1}$$

subject to the dynamic equations, terminal conditions and boundary conditions

$$\begin{aligned} \dot{\boldsymbol{x}} &= f(\boldsymbol{x}, \boldsymbol{u}, t) \\ \boldsymbol{\Psi}[\boldsymbol{x}(t_f), t_f] &= 0 \\ \boldsymbol{x}(t_0) &= 0 \end{aligned}$$

Here $\boldsymbol{x}(t) \in \boldsymbol{R}^n$, $\boldsymbol{u}(t) \in \boldsymbol{R}^l$, and $\boldsymbol{\Psi} \in \boldsymbol{R}^k$. Assume that the functions ϕ, L, and f, respectively, are continuously differentiable with respect to all their arguments. The Hamiltonian function is defined with Lagrange multipliers $\boldsymbol{\lambda}(t) \in \boldsymbol{R}^n$ as

$$H = \boldsymbol{\lambda}^T f + L \tag{2}$$

The minimum principle requires that the control $\boldsymbol{u}$ minimise H:

$$\boldsymbol{u}^* = \underset{\boldsymbol{u}\in\Omega}{\arg\min}\, H(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{u}, t) \tag{3}$$

where Ω is the set of admissible piecewise continuous control values and $\boldsymbol{x}^*$, $\boldsymbol{\lambda}^*$, and $\boldsymbol{u}^*$ are the extremal of the state, costate, and control variables. The state, costate variables

and the Hamiltonian satisfy the following conditions:

$$\dot{\boldsymbol{x}}^T = H_\lambda \tag{4a}$$
$$\dot{\boldsymbol{\lambda}}^T = -H_x \tag{4b}$$
$$\boldsymbol{\lambda}^T(t_f) = \Big[\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{\nu}^T \frac{\partial \Psi}{\partial \boldsymbol{x}}\Big]\Big|_{t=t_f} \tag{4c}$$
$$H(t_f) = -\Big[\frac{\partial \phi}{\partial t} + \boldsymbol{\nu}^T \frac{\partial \Psi}{\partial t}\Big]\Big|_{t=t_f} \tag{4d}$$
$$H_{\boldsymbol{u}} = 0 \tag{4e}$$

where $\boldsymbol{\nu}$ is a constant multiplier vector of the dimension of the constraint Ψ. Control and state inequality constraints are augmented to the Hamiltonian, and additional necessary conditions are obtained as a result. These necessary conditions depending on the type of the state constraint.

3. Problem Formulation

We consider two-link manipulator as developed by Wie, Chuang and Sunkel [6]. The dynamic behaviour of the system is described by the following state equations,

$$\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= \frac{I_2 u_1 - (\alpha I_2 + I_4 \cos x_3) u_2 + I_2 I_4 (x_2 + x_4)^2 \sin x_3 + I_4^2 x_2^2 \sin x_3 \cos x_3}{I_2 I_3 - I_4^2 \cos^2 x_3} \\
\dot{x_3} &= x_4 \\
\dot{x_4} &= [-(I_2 + I_4 \cos x_3)(u_1 + I_4 (x_2 + x_4)^2 \sin x_3) + (\alpha I_2 + I_3 + \\
&\quad (1+\alpha) I_4 \cos x_3) u_2 - I_4 (I_3 + I_4 \cos x_3) x_2^2 \sin x_3] / [I_2 I_3 - I_4^2 \cos^2 x_3]
\end{aligned}$$

where α is a system parameter used to denote the type of torque applied to the second link, $\alpha = 1$ for a joint torque and $\alpha = 0$ for a direct torque, I_1 is the mass moment of inertia of the first link with respect to the shoulder axis scaled by T_{max}, I_2 is the mass moment of inertia of the second link with respect to the elbow axis scaled by T_{max}:

$$\begin{aligned}
I_3 &= I_1 + m_2 L_1^2 / T_{max} \\
I_4 &= m_2 r L_1^2 / T_{max}
\end{aligned}$$

The minimum energy problem is investigated as the performance index

$$\min \mathcal{J}(u) = \int_0^{t_f} \sum_{i=1}^{2} (u_i)^2 dt \tag{5}$$

subject to initial conditions $\boldsymbol{x}_0$ and final conditions $\boldsymbol{x}_{t_f}$.

The final time t_f has to be prescribed in order to obtain useful solution. The control u_i are bounded as follows:

$$|u_i(t)| \leq 1, \quad i = 1, 2 \tag{6}$$

The state variables are constrained by

$$|x_i(t)| \leq x_{i_{max}}, \quad i = 1, \ldots, 4 \tag{7}$$

3.1. Unconstrained problem

The problem is transformed into Mayer problem by introducing variable x_5, where

$$\dot{x_5} = \sum_{i=1}^{2} (u_i)^2, \quad x_5(0) = 0, \quad \min \mathcal{J}(u) = x_5(t_f) \tag{8}$$

Thus the Hamiltonian for the unconstrained problem can be defined by

$$H^{free} = \lambda_{x_1} \dot{x_1} + \lambda_{x_2} \dot{x_2} + \lambda_{x_3} \dot{x_3} + \lambda_{x_4} \dot{x_4} + \lambda_{x_5} \sum_{i=1}^{3} (u_i)^2 \tag{9}$$

The costate equations, defined by $\dot{\boldsymbol{\lambda}}^T = -H_x$

$$\dot{\lambda}_{x_1} = 0 \tag{10}$$

$$\begin{aligned} \dot{\lambda}_{x_2} = & -\lambda_{x_1} - \frac{\lambda_{x_2}(2I_2I_4(x_2 + x_4)\sin x_3 + 2I_4^2 x_2 \sin x_3 \cos x_3)}{I_2 I_3 - I_4^2 \cos^2 x_3} \\ & - \frac{\lambda_{x_4}(-2I_4 x_2 \sin x_3 (I_3 + I_4 \cos x_3)}{I_2 I_3 - I_4^2 \cos^2 x_3} \\ & - \frac{2I_4(x_2 + x_4)(I_2 + I_4 \cos x_3)\sin x_3)}{I_2 I_3 - I_4^2 \cos^2 x_3} \end{aligned} \tag{11}$$

$$\begin{aligned} \dot{\lambda}_{x_3} = & -\frac{1}{I_2 I_3 - I_4^2 \cos^2 x_3} \\ & \Big\{ \lambda_{x_2}(-I_4 \sin x_3 u_2 + I_2 I_4 (x_2 + x_4)^2 \cos x_3 + I_4^2 x_2^2 (\cos^2 x_3 - \sin^2 x_3)) \\ & + \lambda_{x_4}(I_4 \sin x_3 u_1 - (1 + \alpha) I_4 \sin x_3 u_2 - I_4 (I_3 + I_4 \cos x_3) x_2^2 \cos x_3 \\ & - I_4 (I_2 + I_4 \cos x_3)(x_2 + x_4)^2 \cos x_3 - I_4^2 \sin^2 x_3 (x_2^2 + (x_2 + x_4)^2) \Big\} \\ & - \frac{1}{(I_2 I_3 - I_4^2 \cos^2 x_3)^2} \Big\{ 2I_4^2 \cos x_3 \sin x_3 (\lambda_{x_2} \dot{x_2} + \lambda_{x_4} \dot{x_4}) \Big\} \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{\lambda}_{x_4} = & -\frac{\lambda_{x_2}(2I_2 I_4 (x_2 + x_4)\sin x_3)}{I_2 I_3 - I_4^2 \cos^2 x_3} - \lambda_{x_3} \\ & - \frac{\lambda_{x_4}(2I_4 (I_2 + I_4 \cos x_3)(x_2 + x_4)\sin x_3)}{I_2 I_3 - I_4^2 \cos^2 x_3} \end{aligned} \tag{13}$$

$$\dot{\lambda}_{x_5} = 0 \tag{14}$$

The controls can be derived explicitly from $H_{\boldsymbol{u}} = 0$ as follows:

$$u_1 = \frac{I_2(\lambda_4 - \lambda_2) + I_4\lambda_4 \cos x_3}{2\lambda_5(I_2 I_3 - I_4^2 \cos^2 x_3)} \tag{15}$$

$$u_2 = \frac{I_2\alpha(\lambda_2 - \lambda_4) - I_3 I_4 + I_4(\lambda_2 - (1+\alpha)\lambda_4)\cos x_3}{2\lambda_5(I_2 I_3 - I_4^2 \cos^2 x_3)} \tag{16}$$

The transversality condition (eq. [4c]) gives

$$\lambda_5(t_f) = \frac{\partial \mathcal{J}}{\partial x_5(t_f)} = 1 \tag{17}$$

The control u_i are constrained as follows:

$$|u_i(t)| \leq 1, \quad i = 1, 2 \tag{18}$$

The final time t_f is fixed on 0.8 sec. Figure 1 shows that u_1 is directly on the minimum value. Then follows by unconstrained case (see Eq. [15]). Finally u_1 is saturated on the maximum value. While u_2 is unconstrained along the optimal trajectory.

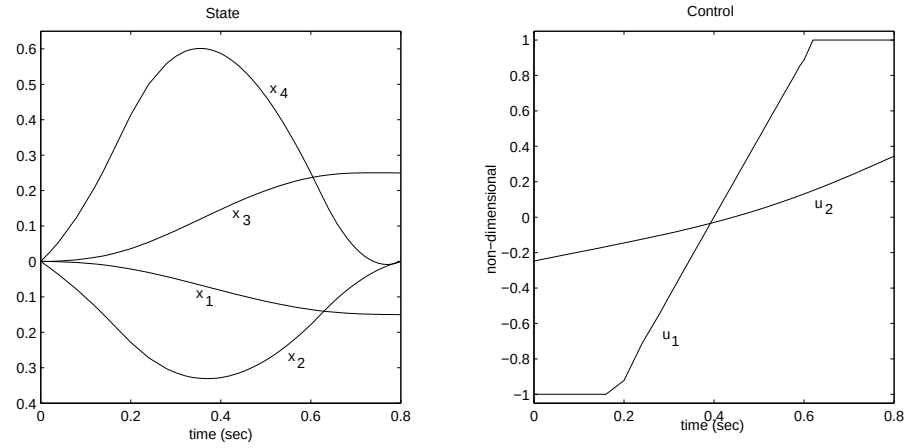

Figure 1: The computatinal results of the unconstrained case

3.2. Constrained problem

The state constraints can be written as

$$S_i := x_i - x_{i_{max}} \leq 0 \tag{19}$$

Consider the following equations:

$$\frac{\partial}{\partial u_i} S_i^{(1)} = 0, \quad \frac{\partial}{\partial u_i} S_i^{(2)} \neq 0, \quad i = 1, 3 \tag{20}$$

and

$$\frac{\partial}{\partial u_i} S_i^{(1)} \neq 0, \quad i = 2, 4 \tag{21}$$

where

$$S_i^{(k)} := \frac{d^k}{dt^k} S_i, \quad i = 1, \ldots, 4 \quad k = 1, 2, \ldots \tag{22}$$

From eq. [20] and [21] we obtain that the state constraints x_1 and x_3 are second order state constraints and the state constraints x_2 and x_4 are first order state constraints. The Hamiltonian for constrained problem becomes

$$H^{cons} = H^{free} + \eta_1 S_1^{(2)} + \eta_2 S_2^{(1)} + \eta_3 S_3^{(2)} + \eta_4 S_4^{(1)} \tag{23}$$

The costate equations can be derived as in the unconstrained case by considering whether the constraints active or not.

4. Numerical Example

This section presents an example for the constrained minimum-energy problem. The initial conditions are $x_0 = [0, 0, 0, 0]$ and the final conditions are $x_{t_f} = [-0.15, 0, 0.25, 0]$. The time t_f is fixed on 0.8 sec and $\alpha = 1$. The state constraints are $|x_i(t)| \leq 0.4, \quad i = 1, \ldots, 4$. The computational results are based on the direct collocation (DIRCOL) by Von Stryk [7].

Figure 2 shows that the state constraint x_4 is active while the other state constraints are not active. When the state constraint x_4 is active the Hamiltonian can be defined by

$$H^{cons} = H^{free} + \eta_4 S_4^{(1)} \tag{24}$$

The control u_1 is saturated directly on the minimum value at the beginning . Then follows by unconstrained case. Finally the control u_1 is saturated on the maximum value. Furthermore the control u_2 is not constrained along the optimal trajectory.

5. Conclusions

The optimal trajectory of the minimum-energy of two-link manipulator is presented. The computational results are based on the direct methods. The advantage of the direct methods is that the user does not have to analyse further into the problem by deriving costate variables, jump conditions or switching structures.

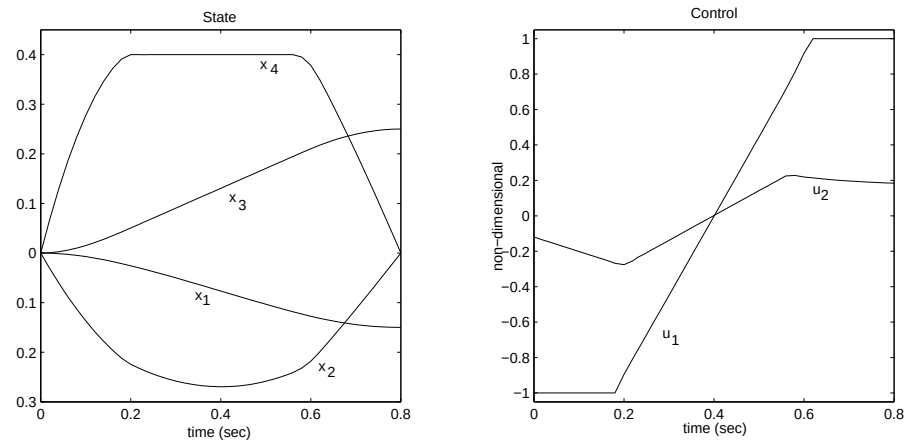

Figure 2: The computational results of the constrained case

The main drawback of the direct methods is that they produce several minima solutions and the solutions are less accurate than the indirect methods. To overcome these problems it is necessary to use the direct method solutions as a starting analysis and initial guesses for the indirect methods.

The main advantage of the indirect methods is that they produce very accurate result. The major difficulties of the indirect methods are that the user must derive the costate variables, jump conditions and switching structures. It is very difficult to define where the jump conditions and switching structures should occur without knowing the direct methods solutions as a priori estimate.

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