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# On Subclass of Bazilevič Function $B_1(\alpha)$, It's Distortion and the Fekete-Szegö Problem

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**Abstract**

In this paper we present the distortion and the Fekete-Szegö problem of subclass of Bazilevič functions, $B_1(\alpha)$. First, we present the result of Singh concerning the sharp value of the coefficients for $B_1(\alpha)$, $|a_2|$, $|a_3|$ and $|a_4|$. Second, we give a solution of the Fekete-Szegö problem, i.e. an estimate of $|a_3 - \mu a_2^2|$ for any real and complex numbers $\mu$ where $a_2$ and $a_3$ are the coefficients of functions $f$ in $B_1(\alpha)$, where $B_1(\alpha)$ is defined by (2), i.e. for each $\alpha > 0$ and for $z \in D$, $Re\ f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} > 0$. These results are sharp for the functions $f_0$ defined by (3) for any real number $\mu$ which satisfies $\mu < (1-\alpha)/2$, or $\mu \geq (4+3\alpha+\alpha^2)/[2(2+\alpha)]$ and for any complex number $\mu$ which satisfies $|3+\alpha-2\mu(2+\alpha)| \geq (1+\alpha)^2$. These results are sharp for the functions $f_1$ defined by (4) for the other real and complex numbers $\mu$. Next, we use similar methods to get estimates for linear expressions involving higher coefficients of function in $B_1(\alpha)$.

## 1. Introduction

In this paper we are concerned with the distortion and the Fekete Szego problems for $B_1(\alpha)$, a subclass of Bazilevič functions $B(\alpha)$. These functions form a subclass of the class of the univalent functions and, at the present time, appear to be the largest subclass defined by an explicit representation. The Bazilevič functions contain many classes of functions that have been extensively studied, e.g. the classes of convex, starlike, spiral-like and close-to-convex functions. A function $f$ is said to be univalent in the unit disc $D = \{z : |z| < 1\}$ if $f$ is analytic in $D$ and if for $z_1, z_2 \in D, f(z_1) = f(z_2)$ if, and only if, $z_1 = z_2$. Since $f$ is analytic in $D$, by McLaurin's theorem we can write $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$. Furthermore, since the derivative of any univalent function does not vanish, without loss of generality, we can normalize the McLaurin series expansion of $f$ so that $a_0 = 0$ and $a_1 = 1$. Thus we may assume that if $f$ is analytic and univalent in $D$, $f$ can be written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

for $z \in D$. We shall denote this class of functions by $S$.

The theory of univalent functions dates from around the beginning of twentieth century, and remains a very active field of research in complex analysis. It could be said that the subject was born with the famous conjecture of Bieberbach in 1916. He showed that if $f \in S$ and is given by (1) then $|a_2| \leq 2$ and proposed his celebrated conjecture that $|a_n| \leq n$ for all $n \geq 2$, with equality for the Koebe function $k$, defined for $z \in D$ by $k(z) = z/(1-z)^2$.

Let $f$ be analytic in the unit disc $D = \{z : |z| < 1\}$, and normalized such that $f(0) = f'(0) - 1 = 0$. Then $f$ is called a Bazilevič function [1]if there exists $g \in St$ such that for each $\alpha > 0$ and $\delta$ real,

$$Re\left[f'(z)\left(\frac{f(z)}{z}\right)^{\alpha+i\delta-1}\left(\frac{g(z)}{z}\right)^{-\alpha}\right] > 0,$$

whenever $z \in D$, where $St$ is the class of normalized starlike univalent functions. We consider the coefficient problem. In 1999 Marjono [6] present a new result about an estimate of the fifth coefficient for functions $f$ in subclass $B_1(\alpha)$, where $B_1(\alpha)$ is defined as follows : Let $f$ be analytic in the unit disc $D$, and normalized. Then $f \in B_1(\alpha)$ if, and only if, for each $\alpha > 0$ and for $z \in D$,

$$Re\ f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} > 0. \tag{2}$$

His result was based on the work of Singh [7] who gave sharps estimates for modulus of the coefficients $a_2, a_3, a_4$. This problem is called as the distortion problem, and people are usually pay more attention to the sharp value. Singh [7] proved the following distortion theorem.

**Theorem 1.1** *Let $f \in B_1(\alpha)$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then*

$$(i) \quad |a_2| \leq \frac{2}{1+\alpha},$$

$$(ii) \quad |a_3| \leq \begin{cases} \frac{2(3+\alpha)}{(2+\alpha)(1+\alpha)^2}, & \text{if } 0 \leq \alpha \leq 1, \\ \frac{2}{2+\alpha}, & \text{if } \alpha \geq 1, \end{cases}$$

$$(iii) \quad |a_4| \leq \begin{cases} \frac{2}{3+\alpha} + \frac{4(1-\alpha)(\alpha^2+3\alpha+5)}{3(2+\alpha)(1+\alpha)^3}, & \text{if } 0 \leq \alpha \leq 1, \\ \frac{2}{3+\alpha}, & \text{if } \alpha \geq 1. \end{cases}$$

*These results are sharp. When $0 \leq \alpha \leq 1$, both inequalities are sharp for $f_0$, which is defined by*

$$f_0(z) = \left( \alpha \int_0^z t^{\alpha-1} \left[ \frac{1+t}{1-t} \right] dt \right)^{1/\alpha}, \tag{3}$$

*and if $\alpha \geq 1$, the second inequality in (ii) is sharp for $f_1$ given by*

$$f_1(z) = \left( \alpha \int_0^z t^{\alpha-1} \left[ \frac{1+t^2}{1-t^2} \right] dt \right)^{1/\alpha}, \tag{4}$$

*the second inequality in (iii) is sharp for $f_2$ given by*

$$f_2(z) = \left( \alpha \int_0^z t^{\alpha-1} \left[ \frac{1+t^3}{1-t^3} \right] dt \right)^{1/\alpha}. \tag{5}$$

Now, we are giving the proof of this theorem which was done by Singh [7]. The basic idea of the proof is comparing coefficients of the series representation of $B_1(\alpha)$ and also considering the properties of functions with positive real parts. For $f \in B_1(\alpha)$, (2) gives

$$f'(z) \left( \frac{f(z)}{z} \right)^{\alpha-1} = p(z), \tag{6}$$

for $z \in D$ and for some $p \in P$, where $P$ is the class of functions with positive real part. Setting $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ and comparing coefficients in (6), we obtain

$$c_1 = (1+\alpha)a_2, \tag{7}$$

$$c_2 = (2+\alpha)a_3 + a_2^2(\alpha-1)(2+\alpha)/2, \tag{8}$$

$$c_3 = (3+\alpha)a_4 + (\alpha-1)(3+\alpha)\left[ a_2 a_3 + \frac{(\alpha-2)a_2^3}{6} \right]. \tag{9}$$

(i) follows immediately from (7) and the fact that $c_n \leq 2$,for the functions with positive real part. We will also use the following lemma which is well known as Carathéodory-Toeplitz inequality.

**Lemma 1.2** [4]. *Let $p \in P$ and $p(z) = 1 + c_1 z + c_2 z^2 + \dots$.*
*Then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|^2}{2}. \tag{10}$$

Eliminating $a_2$ from (7) and (8), we get

$$a_3 = \frac{c_2}{2+\alpha} + \frac{c_1^2(1-\alpha)}{2(1+\alpha)^2}. \tag{11}$$

Thus when $0 \le \alpha \le 1$, (11) and the coefficient of functions with positive real part gives the first inequality in (ii). Both these inequalities are sharp for $f_0$, which is defined by (3). By using a consequence of the Carathéodory-Toeplitz inequality, we can replace $c_2$ from (11) by $c_1^2/2 + \epsilon(2 - |c_1|^2/2)$, where $\epsilon \le 1$. Then applying the triangle inequality, we have

$$|a_3| \le \frac{2}{2+\alpha} + \frac{|c_1|^2}{2(2+\alpha)}(|y| - 1),$$

where $y = (3+\alpha)/(1+\alpha)^2$. Elementary calculation now shows that if $\alpha \ge 1$, then $|y| \le 1$ and so the second inequality in (ii) follows. This inequality is sharp for $f_1$ given by (4).

To prove (iii) eliminating $a_2$ and $a_3$ in (9) we obtain

$$(3+\alpha)a_4 = c_3 + \frac{(1-\alpha)(3+\alpha)}{1+\alpha}\left(\frac{c_1 c_2}{2+\alpha} + \frac{(1-2\alpha)c_1^3}{6(1+\alpha)^2}\right). \tag{12}$$

Now suppose that $0 \le \alpha \le 1/2$. From (12) and using $|c_n| \le 2$, we obtain

$$|a_4| \le \frac{2}{3+\alpha} + \frac{4(1-\alpha)(5+3\alpha+\alpha^2)}{3(1+\alpha)^3(2+\alpha)},$$

which gives the first inequality in (iii).

Next let $1/2 \le \alpha \le 1$. Again, by using a consequence of the Carathéodory-Toeplitz inequality, we can replace $c_2$ in (12) by $c_1^2/2 + \epsilon(2 - |c_1|^2/2)$, where $\epsilon \le 1$. Then from (12), we have

$$|a_4| \le \frac{|c_3|}{3+\alpha} + \frac{|c_1|(1-\alpha)}{1+\alpha}\left\{\frac{|c_1|^2}{2(2+\alpha)}\left(\frac{\alpha^2+3\alpha+5}{3(1+\alpha)^2} - 1\right) + \frac{2}{2+\alpha}\right\}.$$

Using the functions with positive real part, we obtain the first inequality in (iii). This inequality is sharp for $f_0$, which defined by (3). By the simillar method we are doing for $\alpha > 1$ and finally this completes the proof of Theorem 1.

## 2. Known Result of The Fekete-Szegö Problem

A classical theorem of Fekete and Szegö [3] states that for $f \in S$ given by (1),

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3-4\mu, & \text{if } \ \mu \leq 0, \\ 1+2\exp\left(\frac{-2\mu}{1-\mu}\right), & if \ \ 0 \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \ \mu \geq 1, \end{cases}$$

and that this is sharp.

For the subclasses $C,\ St$ and $K$ of convex, starlike and close-to-convex functions respectively, sharp upper bounds for the functional $|a_3 - \mu a_2^2|$ have been obtained for all real $\mu$, e.g. Keogh and Merkes [5] and also Thomas and Darus [2]. In particular for $f \in K$ given by (1), Keogh and Merkes [5] showed that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3-4\mu, & \text{if } \ \mu \leq \frac{1}{3}, \\ \frac{1}{3} + \frac{4}{9\mu}, & \text{if } \ \frac{1}{3} \leq \mu \leq \frac{2}{3}, \\ 1, & \text{if } \ \frac{2}{3} \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \ \mu \geq 1, \end{cases}$$

and that for each $\mu$, there is a function in $K$ such that equality holds.

## 3. Results

We give the solution of the Fekete-Szegö problem for the class $B_1(\alpha)$ for both real and complex parameters $\mu$. The used method was similar to the previous one for the convex and starlike functions.

**Theorem 3.1** [7]. *Let $f \in B_1(\alpha)$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then*

*(i) For any real number $\mu$, we have :*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2}{2+\alpha} + \frac{2(1-\alpha-2\mu)}{(1+\alpha)^2}, & if\ \mu \leq \frac{1-\alpha}{2}, \\ \frac{2}{2+\alpha}, & if\ \frac{1-\alpha}{2} \leq \mu \leq \frac{4+3\alpha+\alpha^2}{2(2+\alpha)}, \\ \frac{2}{2+\alpha} + \frac{4\mu(2+\alpha)-2(4+3\alpha+\alpha^2)}{(2+\alpha)(1+\alpha)^2}, & if\ \mu \geq \frac{4+3\alpha+\alpha^2}{2(2+\alpha)}. \end{cases}$$

*The first and the third inequalities are sharp for the functions $f_0$ defined by (3) The second inequality is sharp for the functions $f_1$ defined by (4)* *(ii) For any complex number $\mu$, we have :*

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{2}{2+\alpha}, & \text{if } |3+\alpha-2\mu(2+\alpha)| \leq (1+\alpha)^2, \\ \frac{2}{2+\alpha}+\frac{2|3+\alpha-2\mu(2+\alpha)|-2(1+\alpha)^2}{(2+\alpha)(1+\alpha)^2}, & \\ \qquad \text{if } |3+\alpha-2\mu(2+\alpha)| \geq (1+\alpha)^2. \end{cases}$$

*The first inequality is sharp for the functions $f_1$ defined by (4) and the second inequality is sharp for the functions $f_0$ defined by (3).*

Now we will give the proof of Theorem 3.1.

*Proof of Theorem 3.1.*
Eliminating $a_2$ and $a_3$ from (7) and (11), we have

$$\begin{aligned} |a_3-\mu a_2^2| &= \left|\frac{c_2}{2+\alpha}+\frac{c_1^2(1-\alpha-2\mu)}{2(1+\alpha)^2}\right| \\ &\leq \left(\frac{|c_2-c_1^2/2|}{2+\alpha}+\frac{|c_1|^2|(1+\alpha)^2+(1-\alpha-2\mu)(2+\alpha)|}{2(2+\alpha)(1+\alpha)^2}\right). \end{aligned}$$

By using the Carathéodory-Toeplitz inequality (10) and the fact that $|c_n| \leq 2$, we obtain

$$\begin{aligned} |a_3-\mu a_2^2| &\leq \left(\frac{2}{2+\alpha}-\frac{|c_1|^2}{2(2+\alpha)}\right)+\frac{|c_1|^2|(1+\alpha)^2+(1-\alpha-2\mu)(2+\alpha)|}{2(2+\alpha)(1+\alpha)^2} \\ &\leq 2\left(\frac{1}{2+\alpha}+\frac{-(1+\alpha)^2+|(1+\alpha)^2+(1-\alpha-2\mu)(2+\alpha)|}{(2+\alpha)(1+\alpha)^2}\right), \end{aligned}$$

i.e.

$$|a_3-\mu a_2^2| \leq 2\left(\frac{1}{2+\alpha}+\frac{|\lambda(\alpha,\mu)|-(1+\alpha)^2}{(2+\alpha)(1+\alpha)^2}\right), \tag{13}$$

where $\lambda(\alpha,\mu)=(1+\alpha)^2+(1-\alpha-2\mu)(2+\alpha)$.

When $\mu$ is a real number, then we have three cases.

**Case (1):** $\lambda(\alpha,\mu) \geq 0$ and $|\lambda(\alpha,\mu)| \geq (1+\alpha)^2$.
It follows that $(1-\alpha-2\mu)(2+\alpha) \geq 0$. Since $(2+\alpha)>0$, we have $(1-\alpha-2\mu) \geq 0$, i.e.

$$\mu \leq \frac{1-\alpha}{2}.$$

Thus (13) becomes

$$|a_3 - \mu a_2^2| \le \frac{2}{2+\alpha} + \frac{2(1-\alpha-2\mu)}{(1+\alpha)^2},$$

and so (i) of Theorem 3.1, provided that $\mu \le (1-\alpha)/2$.

**Case (2):** $\lambda(\alpha,\mu) < 0$ and $|\lambda(\alpha,\mu)| \ge (1+\alpha)^2$.
It follows that $-2(1+\alpha)^2 + (\alpha-1+2\mu)(2+\alpha) \ge 0$. Since $(2+\alpha) > 0$, we have

$$(\alpha - 1 + 2\mu) \ge \frac{2(1+\alpha)^2}{2+\alpha}, \quad \text{i.e.} \quad \mu \ge \frac{4+3\alpha+\alpha^2}{2(2+\alpha)}.$$

Thus (13) becomes

$$|a_3 - \mu a_2^2| \le \frac{2}{2+\alpha} + \frac{4\mu(2+\alpha) - 2(4+3\alpha+\alpha^2)}{(2+\alpha)(1+\alpha)^2},$$

and so (i) of Theorem 3.1, provided that

$$\mu \ge \frac{4+3\alpha+\alpha^2}{2(2+\alpha)}.$$

**Case (3):** $|\lambda(\alpha,\mu)| \le (1+\alpha)^2$.
It follows that $\quad -(1+\alpha)^2 \le \lambda(\alpha,\mu) \le (1+\alpha)^2$, and so

$$0 \le (\alpha - 1 + 2\mu) \le \frac{2(1+\alpha)^2}{2+\alpha}, \quad \text{i.e.} \quad \frac{1-\alpha}{2} \le \mu \le \frac{4+3\alpha+\alpha^2}{2(2+\alpha)}.$$

Thus (13) becomes $|a_3 - \mu a_2^2| \le 2/(2+\alpha)$, and so (i) of Theorem 3.1, provided that

$$\frac{1-\alpha}{2} \le \mu \le \frac{4+3\alpha+\alpha^2}{2(2+\alpha)}.$$

When $\mu$ is a complex number, then by considering the cases $|\lambda(\alpha,\mu)| \le (1+\alpha)^2$ or $|\lambda(\alpha,\mu)| \ge (1+\alpha)^2$, (ii) of Theoremrem 3.1 holds, if
$|3+\alpha-2\mu(2+\alpha)| \le (1+\alpha)^2$ or $|3+\alpha-2\mu(2+\alpha)| \ge (1+\alpha)^2$.

With similar methods as in solving the Fekete-Szegö problem, we also get estimates for linear expressions involving higher coefficients of functions in $B_1(\alpha)$. However, we do not involve the third coefficient of this functions except the second and the fourth. The result is the following.

**Theorem 3.2** *Let* $f \in B_1(\alpha)$ *be given by* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$*, then for* $0 \le \alpha \le 1$*,*

*(i) For any real number* $\mu$*, we have :*

$$\left|a_4 - \tfrac{\mu(1-\alpha)}{6} a_2^3\right| \le \begin{cases} \frac{2}{3+\alpha} + \frac{4(1-\alpha)\{5+3\alpha+\alpha^2-\mu(2+\alpha)\}}{3(1+\alpha)^3(2+\alpha)}, & \text{if } \mu \le 1-2\alpha, \\ \frac{2}{3+\alpha} + \frac{4(1-\alpha)}{(1+\alpha)(2+\alpha)}, & \text{if } 1-2\alpha \le \mu \le 1-2\alpha + \frac{3(1+\alpha)^2}{(2+\alpha)}, \\ \frac{2}{3+\alpha} + \frac{4(1-\alpha)\{\mu(2+\alpha)-(5+3\alpha+\alpha^2)\}}{3(1+\alpha)^3(2+\alpha)}, & \text{if } \mu \ge 1-2\alpha + \frac{3(1+\alpha)^2}{(2+\alpha)}. \end{cases}$$

*The first and the third inequalities are sharp for the functions* $f_0$ *defined by (3). The second inequality is sharp for the functions* $f_2$ *defined by (5).*

*(ii) For any complex number* $\mu$*, we have :*

$$\left|a_4 - \tfrac{\mu(1-\alpha)}{6} a_2^3\right| \le \begin{cases} \frac{2}{3+\alpha} + \frac{4(1-\alpha)}{(1+\alpha)(2+\alpha)}, & \text{if } |1-2\alpha-\mu| \le \frac{3(1+\alpha)^2}{2+\alpha}, \\ \frac{2}{3+\alpha} + \frac{4(1-\alpha)\{|1-2\alpha-\mu|(2+\alpha)-3(1+\alpha)^2\}}{3(1+\alpha)^3(2+\alpha)}, & \text{if } |1-2\alpha-\mu| \ge \frac{3(1+\alpha)^2}{2+\alpha}. \end{cases}$$

*The first inequality is sharp for the functions* $f_2$ *defined by (5) and the second inequality is sharp for the functions* $f_0$ *defined by (3).*

Now, we are proving the Theorem 3.2 by considering the values of $\mu$ into many cases.

*Proof of Theorem 3.2.*

Eliminating $a_2$ and $a_4$ from (7) and (12), we have

$$\left|a_4 - \frac{\mu(1-\alpha)}{6} a_2^3\right| = \left|\frac{c_3}{3+\alpha} + \frac{1-\alpha}{1+\alpha}\left(\frac{c_1 c_2}{2+\alpha} + \frac{(1-2\alpha-\mu)c_1^3}{6(1+\alpha)^2}\right)\right|. \tag{14}$$

By using the Carathéodory-Toeplitz inequality (10) the RHS of (14) becomes

$$= \left|\frac{c_3}{3+\alpha} + \frac{1-\alpha}{1+\alpha}\left\{\frac{c_1^3}{2(2+\alpha)}\left(1 + \frac{(1-2\alpha-\mu)(2+\alpha)}{3(1+\alpha)^2}\right) + \frac{\epsilon c_1}{2+\alpha}\left(2 - \frac{|c_1|^2}{2}\right)\right\}\right|$$

$$\le \frac{|c_3|}{3+\alpha} + \frac{|c_1|(1-\alpha)}{1+\alpha}\left\{\frac{|c_1|^2}{2(2+\alpha)}\left(\left|1 + \frac{(1-2\alpha-\mu)(2+\alpha)}{3(1+\alpha)^2}\right| - 1\right) + \frac{2}{2+\alpha}\right\}, \tag{15}$$

since $|\epsilon| \leq 1$ and $2 - |c_1|^2/2 \geq 0$.

When $\mu$ is a real number, then we have the following cases.

**Case (1):** $1 - 2\alpha - \mu \geq 0$.
By using the fact that $|c_n| \leq 2$, equation (14) becomes

$$\begin{aligned} \left|a_4 - \frac{\mu(1-\alpha)}{6}a_2^3\right| &\leq \frac{2}{3+\alpha} + \frac{4(1-\alpha)}{1+\alpha}\left(\frac{1}{2+\alpha} + \frac{1-2\alpha-\mu}{3(1+\alpha)^2}\right) \\ &= \frac{2}{3+\alpha} + \frac{4(1-\alpha)\{5+3\alpha+\alpha^2-\mu(2+\alpha)\}}{3(1+\alpha)^3(2+\alpha)}, \end{aligned}$$

and so (i) of Theorem 3.2, provided that $\mu \leq 1 - 2\alpha$.

**Case (2):** $\mu \geq 1 - 2\alpha$ and $(1 - 2\alpha - \mu)(2 + \alpha) \geq -3(1 + \alpha)^2$.
It follows that $\mu(2 + \alpha) \leq 3(1 + \alpha)^2 + (1 - 2\alpha)(2 + \alpha)$, i.e.

$$1 - 2\alpha \leq \mu \leq 1 - 2\alpha + \frac{3(1+\alpha)^2}{2+\alpha}.$$

Thus (15) becomes

$$\left|a_4 - \frac{\mu(1-\alpha)}{6}a_2^3\right| \leq \frac{2}{3+\alpha} + \frac{4(1-\alpha)}{(1+\alpha)(2+\alpha)},$$

and so (i) of Theorem 3.2, provided that

$$1 - 2\alpha \leq \mu \leq 1 - 2\alpha + \frac{3(1+\alpha)^2}{2+\alpha}.$$

**Case (3):** $(1 - 2\alpha - \mu)(2 + \alpha) \leq -3(1 + \alpha)^2$.
It follows that $\mu(2 + \alpha) \geq 3(1 + \alpha)^2 + (1 - 2\alpha)(2 + \alpha)$, i.e.

$$\mu \geq 1 - 2\alpha + \frac{3(1+\alpha)^2}{2+\alpha}.$$

Thus (15) becomes

$$\begin{aligned} &\leq \frac{2}{3+\alpha} + \frac{2(1-\alpha)}{1+\alpha}\left\{\frac{2}{2+\alpha}\left(\frac{(2\alpha-1+\mu)(2+\alpha)}{3(1+\alpha)^2} - 2\right) + \frac{2}{2+\alpha}\right\} \\ &= \frac{2}{3+\alpha} + \frac{4(1-\alpha)\{(2\alpha-1+\mu)(2+\alpha) - 3(1+\alpha)^2\}}{3(1+\alpha)^3(2+\alpha)} \\ &= \frac{2}{3+\alpha} + \frac{4(1-\alpha)\{\mu(2+\alpha) - (5+3\alpha+\alpha^2)\}}{3(1+\alpha)^3(2+\alpha)}, \end{aligned}$$

and so (i) of Theorem 3.2, provided that

$$\mu \geq 1 - 2\alpha + \frac{3(1+\alpha)^2}{2+\alpha}.$$

When $\mu$ is a complex number, then by considering the cases

$$\left|\frac{(1-2\alpha-\mu)(2+\alpha)}{3(1+\alpha)^2}\right| \leq 1 \quad \text{or} \quad \left|\frac{(1-2\alpha-\mu)(2+\alpha)}{3(1+\alpha)^2}\right| \geq 1,$$

(ii) of Theorem 3.2 holds, if we require for $\mu$

$$|1-2\alpha-\mu| \leq \frac{3(1-\alpha)^2}{2+\alpha},$$

or

$$|1-2\alpha-\mu| \geq \frac{3(1-\alpha)^2}{2+\alpha}.$$

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