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On the Boundedness of a Generalized Fractional Integral on Generalized Morrey Spaces

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Abstract

In this paper we extend Nakai's result on the boundedness of a generalized fractional integral operator from a generalized Morrey space to another generalized Morrey or Campanato space.

keywords: *Generalized fractional integrals, generalized Morrey spaces, generalized Campanato spaces*

1. Introduction and Main results

For a given function $\rho : (0,\infty) \longrightarrow (0,\infty)$, let $\mathcal{T}_\rho$ be the generalized fractional integral operator, given by

$$\mathcal{T}_\rho f(x) = \int_{\mathbf{R}^n} \frac{f(y)\rho(|x-y|)}{|x-y|^n} dy,$$

and put

$$\widetilde{\mathcal{T}}_\rho f(x) = \int_{\mathbf{R}^n} f(y) \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_o}(y))}{|y|^n} \right) dy,$$

the modified version of $\mathcal{T}_\rho$, where B_o is the unit ball about the origin, and χ_{B_o} is the characteristic function of B_o.

In [4], Nakai proved the boundedness of the operators $\widetilde{\mathcal{T}}_\rho$ and $\mathcal{T}_\rho$ from a generalized Morrey space $\mathcal{M}_{1,\phi}$ to another generalized Morrey space $\mathcal{M}_{1,\psi}$ or generalized Campanato space $\mathcal{L}_{1,\psi}$. More precisely, he proved that

$$\|\mathcal{T}_\rho f\|_{\mathcal{M}_{1,\psi}} \leq C\|f\|_{\mathcal{M}_{1,\phi}} \qquad \text{and} \qquad \|\widetilde{\mathcal{T}}_\rho f\|_{\mathcal{L}_{1,\psi}} \leq C\|f\|_{\mathcal{M}_{1,\phi}},$$

where $C > 0$, with some appropriate conditions on ρ, ϕ and ψ. Using the techniques developed by Kurata *et.al.* [1], we investigate the boundedness of these operators from generalized Morrey spaces $\mathcal{M}_{p,\phi}$ to generalized Morrey spaces $\mathcal{M}_{p,\psi}$ or generalized Campanato spaces $\mathcal{L}_{p,\psi}$ for $1 < p < \infty$.

The generalized Morrey and Campanato spaces are defined as follows. For a given function $\phi : (0,\infty) \longrightarrow (0,\infty)$, and $1 < p < \infty$, let

$$\|f\|_{\mathcal{M}_{p,\phi}} = \sup_B \frac{1}{\phi(B)}\left(\frac{1}{|B|}\int_B |f(y)|^p dy\right)^{\frac{1}{p}},$$

and

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)}\left(\frac{1}{|B|}\int_B |f(y) - f_B|^p dy\right)^{\frac{1}{p}},$$

where the supremum is taken over all open balls $B = B(a,r)$ in $\mathbf{R}^n$, $|B|$ is the Lebesgue measure of B in $\mathbf{R}^n$, $\phi(B) = \phi(r)$, and $f_B = \frac{1}{|B|}\int_B f(y)dy$. We define the Morrey space $\mathcal{M}_{p,\phi}$ by

$$\mathcal{M}_{p,\phi} = \{f \in L^p_{loc}(\mathbf{R}^n) : \|f\|_{\mathcal{M}_{p,\phi}} < \infty\},$$

and the Campanato space $\mathcal{L}_{p,\phi}$ by

$$\mathcal{L}_{p,\phi} = \{f \in L^p_{loc}(\mathbf{R}^n) : \|f\|_{\mathcal{L}_{p,\phi}} < \infty\}.$$

Our results are the following:

Theorem 1.1 *If $\rho, \phi, \psi : (0,\infty) \longrightarrow (0,\infty)$ satisfying the conditions below :*

$$\frac{1}{2} \leq \frac{t}{r} \leq 2 \Rightarrow \frac{1}{A_1} \leq \frac{\phi(t)}{\phi(r)} \leq A_1, \quad \textit{and} \quad \frac{1}{A_2} \leq \frac{\rho(t)}{\rho(r)} \leq A_2 \tag{1}$$

$$\int_0^1 \frac{\rho(t)}{t}dt < \infty, \textit{and for all} r > 0, \textit{we have} \int_r^\infty \frac{\phi(t)^p}{t}dt \leq A_3\phi(r)^p, \tag{2}$$

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \leq A_4\psi(r), \textit{for all} r > 0, \tag{3}$$

where $A_i > 0$ are independent of $t, r > 0$, then for each $1 < p < \infty$ there exists $C_p > 0$ such that

$$\|\mathcal{T}_\rho f\|_{\mathcal{M}_{p,\psi}} \leq C_p \|f\|_{\mathcal{M}_{p,\phi}}.$$

Theorem 1.2 *If $\rho, \phi, \psi : (0, \infty) \longrightarrow (0, \infty)$ satisfying the conditions below*

$$\frac{1}{2} \leq \frac{t}{r} \leq 2 \Rightarrow \frac{1}{A_1} \leq \frac{\phi(t)}{\phi(r)} \leq A_1, \quad and \quad \frac{1}{A_2} \leq \frac{\rho(t)}{\rho(r)} \leq A_2 \tag{4}$$

$$\int_0^1 \frac{\rho(t)}{t} dt < \infty, \text{and for all} r > 0, \text{we have} \int_r^\infty \frac{\phi(t)^p}{t} dt \leq A_3 \phi(r)^p, \tag{5}$$

$$\left|\frac{\rho(r)}{r^n} - \frac{\rho(t)}{t^n}\right| \leq A_4 |r - t| \frac{\rho(r)}{r^{n+1}}, \quad for \quad \frac{1}{2} \leq \frac{t}{r} \leq 2, \tag{6}$$

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + r \int_r^\infty \frac{\rho(t)\phi(t)}{t^2} dt \leq A_5 \psi(r), \text{for all} r > 0, \tag{7}$$

where $A_i > 0$ are independent of $t, r > 0$, then for each $1 < p < \infty$ there exists $C_p > 0$ such that

$$\|\widetilde{T}_\rho f\|_{\mathcal{L}_{p,\psi}} \leq C_p \|f\|_{\mathcal{M}_{p,\phi}}.$$

2. Proof of the Theorems

To prove the theorems, we shall use the following result of Nakai [2] (in a slightly modified version) about the boundedness of the standard maximal function Mf on a generalized Morrey space $\mathcal{M}_{p,\phi}$. The standard maximal function Mf is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \quad x \in \mathbf{R}^n,$$

where the supremum is taken over all open balls B containing x.

Theorem 2.1 *(Nakai).If $\phi : (0, \infty) \longrightarrow (0, \infty)$ satisfying the conditions below :*

(a). $\frac{1}{2} \leq \frac{t}{r} \leq 2 \Rightarrow \frac{1}{A_1} \leq \frac{\phi(t)}{\phi(r)} \leq A_1$,

(b). $\int_r^\infty \frac{\phi(t)^p}{t} dt \leq A_2 \phi(r)^p$, *for all* $r > 0$,

where $A_i > 0$ are independent of $t, r > 0$, then for each $1 < p < \infty$ there exists $C_p > 0$ such that

$$\|Mf\|_{\mathcal{M}_{p,\phi}} \leq C_p \|f\|_{\mathcal{M}_{p,\phi}}.$$

From now on, C and C_p will denote positive constants, which may vary from line to line. In general, these constants depend on n.

Proof of Theorem 1.1
For $x \in \mathbf{R}^n$, and $r > 0$, write

$$\mathcal{T}_\rho f(x) = \int_{|x-y|<r} \frac{f(y)\rho(|x-y|)}{|x-y|^n} dy + \int_{|x-y|\geq r} \frac{f(y)\rho(|x-y|)}{|x-y|^n} dy = I_1(x) + I_2(x).$$

Note that, for $t \in [2^k r, 2^{k+1} r]$, there exist constants $C_i > 0$ such that

$$\rho(2^k r) \leq C_1 \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t} dt$$

and

$$\rho(2^k r)\phi(2^k r) \leq C_2 \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)\phi(t)}{t} dt.$$

So, we have

$$\begin{aligned}
|I_1(x)| &\leq \int_{|x-y|<r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n} dy \\
&\leq \sum_{k=-\infty}^{-1} \int_{2^k r \leq |x-y| < 2^{k+1} r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n} dy \\
&\leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^k r)}{(2^k r)^n} \int_{|x-y|<2^{k+1} r} |f(y)| dy \\
&\leq C \sum_{k=-\infty}^{-1} \rho(2^k r) Mf(x) \\
&\leq C Mf(x) \sum_{k=-\infty}^{-1} \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t} dy \\
&\leq C Mf(x) \int_0^r \frac{\rho(t)}{t} dy \\
&\leq C \frac{\psi(r)}{\phi(r)} Mf(x).
\end{aligned}$$

Meanwhile,

$$
\begin{aligned}
|I_2(x)| &\le \int_{|x-y|\ge r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n}dy \\
&\le \sum_{k=0}^{\infty} \int_{2^k r\le|x-y|<2^{k+1}r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n}dy \\
&\le C\sum_{k=0}^{\infty} \frac{\rho(2^{k+1}r)}{(2^k r)^n} \int_{|x-y|<2^{k+1}r} |f(y)|dy \\
&\le C\sum_{k=0}^{\infty} \rho(2^{k+1}r)\phi(2^{k+1}r)\|f\|_{\mathcal{M}_{p,\phi}} \\
&\le C\|f\|_{\mathcal{M}_{p,\phi}} \sum_{k=0}^{\infty} \int_{2^{k+1}r}^{2^{k+2}r} \frac{\phi(t)\rho(t)}{t}dt \\
&\le C\|f\|_{\mathcal{M}_{p,\phi}} \int_r^{\infty} \frac{\phi(t)\rho(t)}{t}dt \\
&\le C\psi(r)\|f\|_{\mathcal{M}_{p,\phi}}.
\end{aligned}
$$

Now, for $1\le p<\infty$, we have

$$|\mathcal{T}_\rho f(x)|^p \le 2^{p-1}(|I_1(x)|^p + |I_2(x)|^p),$$

and by Nakai's Theorem, we have for all balls $B=B(a,r)$

$$\frac{1}{\psi(r)^p|B|}\int_B |I_1(x)|^p dx \le \frac{C}{\phi(r)^p|B|}\int_B Mf(x)^p dx \le C\|Mf\|^p_{\mathcal{M}_{p,\phi}} \le C_p \,\|f\|^p_{\mathcal{M}_{p,\phi}},$$

and

$$\frac{1}{\psi(r)^p|B|}\int_B |I_2(x)|^p dx \le C\|f\|^p_{\mathcal{M}_{p,\phi}}.$$

Combining the two estimates, we obtain

$$\frac{1}{\psi(r)^p|B|}\int_B |\mathcal{T}_\rho f(x)|^p dx \le C_p \,\|f\|^p_{\mathcal{M}_{p,\phi}},$$

and the result follows. □

Proof of Theorem 1.2

Let $\widetilde{B}=B(a,2r)$. For $x\in B=B(a,r)$, we have

$$\widetilde{\mathcal{T}}_\rho f(x) - C_B = E^1_B(x) + E^2_B(x),$$

where

$$C_B = \int_{\mathbf{R}^n} f(y)\left(\frac{\rho(|a-y|)(1-\chi_{\widetilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_o}(y))}{|y|^n}\right)dy,$$

$$E_B^1(x) = \int_{\widetilde{B}} f(y) \frac{\rho(|x-y|)}{|x-y|^n} dy,$$

and

$$E_B^2(x) = \int_{\widetilde{B}^c} f(y) \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right) dy.$$

From (6), we have

$$|C_B| \le C \left(\int_{|a-y|<k} |f(y)| dy + |a| \int_{|a-y|\ge k} |f(y)| \frac{\rho(|a-y|)}{|a-y|^{n+1}} dy \right),$$

where $k = \max(2|a|, 2r)$, and so we know that C_B is finite for every ball $B = B(a,r)$.

With the same technique as in the proof of the previous theorem, we have

$$\begin{aligned} |E_B^1(x)| &\le \int_{|a-y|<2r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n} dy \\ &\le \int_{|x-y|<3r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n} dy \\ &\le CMf(x) \int_0^{3r} \frac{\rho(t)}{t} dt \\ &\le CMf(x) \int_0^{r} \frac{\rho(t)}{t} dt, \end{aligned}$$

and by (6)

$$\begin{aligned} |E_B^2(x)| &\le \int_{|a-y|\ge 2r} |f(y)| \left| \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right| dy \\ &\le C|x-a| \int_{|a-y|\ge 2r} |f(y)| \frac{\rho(|a-y|)}{|a-y|^{n+1}} dy \\ &\le C\|f\|_{\mathcal{M}_{p,\phi}} r \int_r^\infty \frac{\rho(t)\phi(t)}{t^2} dt, \end{aligned}$$

and the result follows as before. □

3. Remark

We also suspect that $\widetilde{T}_\rho$, the modified version of T_ρ, is bounded from $\mathcal{L}_{p,\phi}$ to $\mathcal{L}_{p,\psi}$ under the same hypothesis on ρ, ϕ and ψ as in Theorem 1.2. However, we have not obtained the proof and the research in this direction is still ongoing.

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