Disturbance Rejection Problem with Stability By Static Output Feedback Of Linear Continuous Time System

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Abstract

Disturbance rejection problem with stability by static output feedback of linear-time invariant continuous time system is solvable if there is found a static output feedback control law, $u(t) = Ky(t)$ (if possible), such that disturbance $q(t)$ has no influence in controlled output $z(t)$. So, it is needed the necessary and sufficient condition disturbance rejection problem is solvable. By using the definition and characteristics of (A, B)-invariant subspace, and (C, A)-invariant subspace, then it will be find the necessary and sufficient condition disturbance rejection problem of that system will be solved if and only if maximal element of a set of (C, A)-invariant is an (A, B)-invariant subspace that internally stabilizable and externally stabilizable.

Key Words: *Linear system; Disturbance rejection problem; (A, B)-invariant subspace; (C, A)-invariant subspace*

1. Introduction

In system theory, there are many system models. Once is a linear, continuous-time system described by:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, $y \in R^h$ is the output. $A \in R^n$ is a matrix, $B \in R^{n \times m}$ is a matrix, and $C \in R^{h \times n}$ is a matrix. If that system has a disturbance, we can make the model of the system likes:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Eq(t) \\
y(t) &= Cx(t)
\end{aligned}
\tag{1}
$$

where $q \in R^q$ is the disturbance input, and $E \in R^{n \times q}$ is a matrix. When there is a disturbance in a system, it may cause deviation between the expecting result and the obtained output. Therefore, it needs a feedback control which becomes control rule which gives information from output deviation and entry that information as a new input, then all the deviations from the expecting result can be corrected. A solution of Disturbance Rejection Problem with stability by Static Input Feedback, $u(t) = Fx(t)$ has been recently proposed in [9]. When the state vector is not available for measurement, [3] are led to consider the Disturbance Rejection Problem by Static Output Feedback, $u(t) = Ky(t)$ with the system is:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Eq(t) \\
y(t) &= Cx(t) \\
z(t) &= Dx(t)
\end{aligned}
\tag{2}
$$

where $y \in R^h$ is measured output $z \in R^z$ is the controlled output, and $D \in R^{z \times n}$ is a matrix . For next, the system in 2 will be called (A, B, C, D, E) system. In this paper will be studied the solution of Disturbance Rejection with Stability by Output Static Feedback related to the system with the system class is:

$$
\gamma^* \subset Ker(D) + S_*
$$

Where $\gamma^* =$ maximal (A, B) invariant subspace contained in Ker (D), and $S_* =$ minimal (C, A) invariant subspace containing Image (E).

2. Preliminaries

Consider (A, B, C, D, E) system, if we give Static Output Feedback, $u(t) = Ky(t)$, where $K \in R^{m \times h}$ is a matrix, then system in (1) becomes:

$$\dot{x}(t) = Ax(t) + BKy(t) + Eq(t)$$
$$\dot{x}(t) = Ax(t) + BKCx(t) + Eq(t)$$

The solution of the equation above by using linear differential equation if $x(0) = x_0$ is:

$$x(t) = x_0 e^{(A+BKC)t} + \int_0^t e^{(A+BKC)t} Eq(s)ds$$

$$z(t) = Dx_0 e^{(A+BKC)t} + D \int_0^t e^{(A+BKC)t} Eq(s)ds$$

The goal of the Disturbance Rejection Problem by Static Output Feedback is to determine, if possible, a static output feedback control law, $u(t) = Ky(t)$, such that the transfer matrix from disturbance q to controlled output z is null, or:

$$D \int_0^t e^{(A+BKC)t} Eq(s)ds = 0$$

Invariant Subspaces

(A, B)-invariant subspace and (C, A)-invariant are basic in the study of this problem. In this section, we bring definition and characteristic both of them.

Definition 2.1 *[3] Consider the pair (A, B) related to the system (1). A subspace $\gamma \subset R^n$ is said to be (A, B) invariant if $A\gamma \subset \gamma + B$.*

We can show that, equivalently, $\gamma \subset R^n$ is (A, B) invariant if and only if there exist a matrix $F \in R^{m \times n}$ such that $(A + BF)\gamma \subset \gamma$.

We can also show that the set of all (A, B)-invariant subspace contained in Ker (D) has a maximal element:

$\gamma^* = $ maximal (A, B)-invariant subspace contained in Ker (D).

Definition 2.2 *[3] Consider the pair (C, A) related to the system (1). A subspace $S \subset R^n$ is said to be (C, A)-invariant if $A(S \cap Ker(C)) \subset S$.*

Equivalently, S is (C, A)-invariant if and only if there exists a matrix $G \in R^{n \times h}$ such that $(A+GC)S \subset S$.

The set of all (C, A)-invariant subspaces containing Im (E) which is defined by $\xi(C, A; Im(E)$ has a minimal element:

$\qquad S_* = $ minimal (C, A)-invariant subspaces containing Im (E).

From the above definition we can prove this theorem:

Theorem 2.3 *[1] (C, A)-invariant is closed under subspace addition.*

Gathering definition and characteristic of (A, B) invariant and (C, A)-invariant, we can make a subspace V which is (A, B) invariant and (C, A)-invariant become a new form of invariant subspace that will be explained in this following theorem:

Theorem 2.4 *[1] Let $A \in R^n$ is a matrix, $B \in R^{n \times m}$ is a matrix, $C \in R^{h \times n}$ is a matrix and $K \in R^{m \times h}$ is a matrix. A subspace V is $(A + BKC)$-invariant if and only if V is (A, B)-invariant and (C, A)-invariant.*

Internally Stabilizable and Externally Stabilizable

An invariant subspace of a matrix $A \in R^n$ always has linear transformation. From that, we can classification an invariant subspace becoming internally stabilizable or externally stabilizable, which will be explained in these following theorem and definition:

Theorem 2.5 *[1] Let $A \in R^n$ is a matrix, and subspace $J \in R^n$ is an A-invariant, then there exist linear transformation T such that*

$$A' = T^{-1}AT = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix} \tag{3}$$

with $A'_{11} \in R^{h \times h}$ is a matrix and dimension of $J = h$.

Definition 2.6 *[1] Let subspace $J \in R^n$ is an A-invariant.*

1. *J is called internally stabilizable if the differences eigen value of A'_{11}, denoted by $\lambda_1, \lambda_2, ..., \lambda_k$ $(k \leq h)$ implies $Re(\lambda_i) < 0$ for $i = 1, 2, ..., k$*

2. *J is called externally stabilizable if the differences eigen value of A'_{22}, denoted by $\lambda_1, \lambda_2, ..., \lambda_k$ $(k \leq (n - h))$ implies $Re(\lambda_i) < 0$ for $i = 1, 2, ..., k$*

We can use this following theorem to look for determinant of A' in (3).

Theorem 2.7 *[1] Let $A \in R^n$ is a matrix, then det $(A') = $ det $(A) = $ det (A'_{11}). det (A'_{22}).*

Consider now the following class of subspaces

$$\langle A + BKC|\mathrm{Im}E\rangle = \mathrm{Im}(E) + (A + BKC)\mathrm{Im}(E) + \cdots + (A + BKC)^{n-1}\mathrm{Im}(E)$$

where $(A + BKC)^{n-1}$ is composition function as n-1 times.

Definition 2.8 *[9] Disturbance rejection problem at (A, B, C, D, E) system with stability by static output feedback can be solved if there exist a matrix $K \in R^{m \times h}$ such that $\langle A + BKC|ImE\rangle \subset Ker(D)$ and $\sigma(A+BKC) \subset C_g$ where $\sigma(A+BKC)$ represents the set of (A + BKC)'s eigen value, and $C_g = \{s; s \in C, Re(\lambda_s) < 0\}$*

Class of System: Self Hidden
In this paper, we want to establish the geometric condition which defined the class of system. The class is satisfying condition that is explained in this following definition:

Definition 2.9 *[3] $S \in \xi(C, A; Im(E)$ is said to be self hidden with respect to Im(E) if $S \subset S_* + Ker(C)$*

3. Discussion

The preceding Theorem establishes necessary and sufficient condition for existence of static output feedback which solves Disturbance Rejection with Stability by Output Static Feedback with the system class is $\gamma^* \subset Ker(C) + S_*$.

The first theorem that we have to prove is showing that there is a subspace which become (A, B)-invariant and also (C, A) invariant likes in this following theorem

Theorem 3.1 $\langle A + BKC|ImE\rangle$ *is (A, B) and (C, A)-invariant.*

Proof:
For arbitrary $h \in (A + BKC)\langle A + BKC|ImE\rangle$, we have $h = (A + BKC) x$, where $x \in \langle A + BKC|ImE\rangle$. Then there exist $w_1, w_2, ..., w_n \in Im(E)$ such that

$$\begin{aligned}
x &= w_1 + (A + BKC)w_2 + ... + (A + BKC)^{n-1}w_n \\
h &= -p_n(w_n) + (A + BKC)(w_1 - p_{n-1}w_n) + (A + BKC)^2(w_2 - p_{n-2}w_n) \\
&\quad + \cdots + (A + BKC)^{n-1}(w_{n-1} - p_1w_n)
\end{aligned}$$

Since Im (E) subspace R^n, we get

$$-p_n(w_n), (w_1 - p_{n-1}w_n), (w_2 - p_{n-2}w_n), ..., (w_{n-1} - p_1w_n) \in Im(E)$$

Therefore we have $(A + BKC)\langle A + BKC|ImE\rangle \subset \langle A + BKC|ImE\rangle$

From Definition 2.1 and Theorem 2.4, $\langle A + BKC|ImE\rangle$ is (A, B) and (C, A)-invariant. $\square$

Before we prove the last theorem, we need to find necessary and sufficient condition of disturbance rejection problem which not contain a system class which is included in this following theorem:

Theorem 3.2 *Disturbance rejection problem at (A, B, C, D, E) system with stability by static output feedback can be solved if and only if there exist V subspace and a matrix $K \in R^{mxh}$ such that*

1. *$Im(E) \subset V \subset Ker(D)$*

2. *$(A + BKC)V \subset V$*

3. *$\sigma(A + BKC) \subset C_g$*

Proof:
Necessity:

1. By using Definition 2.8 there exists a matrix $K \in R^{mxh}$ such that $\langle A + BKC|ImE\rangle \subset Ker(D)$ and $\sigma(A + BKC) \subset C_g$.

   We take $V = \langle A+BKC|ImE\rangle$. For arbitrary $p \in Im(E)$, there exists $z \in R^q$ such that $p = E(z)$.

   $p = E(z) = E(z) + (A + BKC)\overline{0} + (A + BKC)^2\,\overline{0} + ... + (A + BKC)^{n-1}\overline{0}$. Because $E(0) = $, then $p \in \langle A + BKC|ImE\rangle$. So Im $(E) \subset \langle A + BKC|ImE\rangle = V$. By Definition 2.8 we get $\langle A + BKC|ImE\rangle \subset$ Ker (D), so $Im(E) \subset V \subset Ker(D)$.

2. From the Theorem 3.1, we get

   $(A + BKC)(\langle A + BKC|ImE\rangle = V) (\subset \langle A + BKC|ImE\rangle = V)$

3. from the Definition 2.8 we get $\sigma(A + BKC) \subset C_g$

   Sufficiency: Because Im $(E) \subset V$, then

$$\left.\begin{array}{l} Im(E) + (A + BKC)Im(E)+ \\ (A + BKC)^2 Im(E) + ...+ \\ (A + BKC)^{n-1}Im(E) \end{array}\right\} \subset \left\{\begin{array}{l} V + (A + BKC)V+ \\ (A + BKC)^2 V + ...+ \\ (A + BKC)^{n-1}V \end{array}\right.$$

or

$$\langle A + BKC|ImE\rangle \subset \langle A + BKC|V\rangle \tag{4}$$

For arbitrary $g \in \langle A + BKC | ImE \rangle$, there exist $v_1, v_2, v_3, ..., v_n \in V$ such that $g = v_1 + (A + BKC)v_2 + (A + BKC)^2 v_3 + ... + (A + BKC)^{n-1} v_n$. Because $(A + BKC)V \subset V$, then there exist $w_1, w_2, w_3, ..., w_n \in V$ such that $g = v_1 + w_1 + w_2 + w_3 + ... + w_n \in V$. So

$$\langle A + BKC | V \rangle \subset V. \tag{5}$$

Next, let $v \in V$. Because V is a subspace and $(A + BKC)$ is linear transformation, there exists $0 \in V$, so v can be wrote as:

$v = v + (A + BKC)0 + (A + BKC)^2 0 + ... + (A + BKC)^{n-1} 0.$

So $v \in \langle \text{A+BKC} | \text{V} \rangle$ or

$$V \subset \langle A + BKC | V \rangle \tag{6}$$

From (4), (5), and (6) we get

$\langle A + BKC | ImE \rangle \subset \langle A + BKC | V \rangle = V \subset \text{Ker}(D)$

then $\langle A + BKC | Im(E) \rangle \subset \text{Ker}(D)$, or disturbance rejection problem at(A, B, C, D, E) system with stability by static output feedback can be solved. $\square$

Suppose $\Psi (C, A; \text{Im}(E))$ represent the set of (C, A)-invariant subspace containing Im (E) which are self hidden or we can wrote as:

$\Psi(C, A; \text{Im}(E)) = \{S; (A+GC)S \subset S, S \supset \text{Im}(E) \text{ and } S \subset \text{Ker}(C) + S_*\}$

Based on Theorem 2.3 we find that (C, A)-invariant is closed under subspace addition. We also can prove that $(\Psi(C, A; \text{Im}(E))$ is closed under subspace addition

Suppose $\Gamma(C, A; \text{Im}(E))$ represent the set of (C, A)-invariant subspace containing Im (E) and contained in γ^* or it is can be wrote as:

$\Gamma(C, A; \text{Im}(E)) = \{S: S \in \xi(C, A; Im(E) \text{ and } S \subset \gamma^*\}$

Based on above sentences we find that (C, A) invariant is closed under subspace addition.

We also can prove if $\gamma^* \subset Ker(C) + S_*$ implies $\Gamma(C, A; \text{Im}(E))$ is closed under subspace addition.

Based on Definition 2.2, we find that set of all (C, A)-invariant subspaces containing Im (E) has a minimal element. In this section, we also can prove that $\Gamma(C, A; \text{Im}(E))$ also has a maximal element if $\gamma^* \subset Ker(C) + S_*$ which is:

$$S^* = max S \in \Gamma(C, A; Im(E))$$

The last theorem we need to prove is necessary and sufficient condition of disturbance rejection problem at (A, B, C, D, E) system with the system class is $\gamma^* \subset Ker(C) + S_*$ with stability by static output feedback can be solved which is mentioned in following theorem:

Theorem 3.3 *Disturbance rejection problem at (A, B, C, D, E) system with the system class is $\gamma^* \subset Ker(C) + S_*$ with stability by static output feedback can be solved if and only if S^* is an (A, B)-invariant subspace that internally stabilizable and externally stabilizable.*

Proof:
Necessity: There exists matrix $K \in R^{m \times h}$ and subspace S implies $S_* < S < \gamma^*$ and $(A+BKC)S \subset S$. This is guarantying that $\Gamma(C, A; \text{Im}(E))$ is not empty, then there exist maximal element S^*. Because $S \subset S^*$ and S^* is an (C, A)-invariant, then $(A + BKC)S \subset (A + BKC)S^* \subset S^*$. It follow based on Theorem 2.4 that S^* is an (A, B)-invariant.

Because $\sigma(A + BKC) \subset C_g$, then based on Definition 2.8, $\text{Re}(\lambda_i) < 0$ for $i = 1, 2, \ldots, k$ $(k \leq n)$. Suppose $A + BKC = M$, then by Theorem 2.4 there exist T^{-1} and T implies

$$M' = T^{-1}MT = \begin{pmatrix} M'_{11} & M'_{12} \\ 0 & M'_{22} \end{pmatrix} \tag{7}$$

Therefore,

$$
\begin{aligned}
det(\lambda I - M') &= det(\lambda I - M) \\
&= det\begin{pmatrix} \lambda I_h - M'_{11} & -M'_{12} \\ 0 & \lambda I_{n-h} - M'_{22} \end{pmatrix} \\
&= det(\lambda I_h - M'_{11}) \times det(\lambda I_{n-h} - M'_{22})
\end{aligned} \tag{8}
$$

Because $\text{Re}(\lambda_i) < 0$ for $i = 1, 2, \ldots, k$ $(k \leq n)$, then $\text{Re}(\lambda_i) < 0$ for $i = 1, 2, \ldots, k$ $(k \leq h)$ and $\text{Re}(\lambda_i) < 0$ for $i = 1, 2, \ldots, k$ $(k \leq (n-h))$, therefore $\text{Re}(\lambda_i) < 0$ for $i = 1, 2, \ldots, k$ $(k \leq h))$ in sub matrix M'_{11} or $((A + BKC)'_{11})$ and $\text{Re}(\lambda_i) < 0$ for $i = 1, 2, \ldots, k$ $(k \leq (n-h))$ in sub matrix M'_{22} or $((A + BKC)'_{22})$.

It follows based on Definition 2.6, we get S^* is a subspace that internal stabilizable and externally stabilizable. Therefore S^* is an (A, B)-invariant subspace that internally stabilizable and externally stabilizable.

Sufficiency: Suppose S^* is an (A, B)-invariant. Because S^* maximal element in $S \in \Gamma(C, A; Im(E))$ then $\text{Im}(E) \subset S_* \subset S \subset S^* \subset \gamma^* \subset Ker(D)$

or Im $(E) \subset S^* \subset Ker(\mathrm{D})$.

Because S^* is an (A, B)-invariant and (C, A)-invariant then based on Theorem 2.4 S^* is an $(A+BKC)$-invariant, which is implies based on Definition 2.1 $(A+BKC)S^* \subset S^*$.

Because S^* is an (A+BKC)-invariant that internally stabilizable then sub matrix (M'_{11}) in (7) implies $\mathrm{Re}(\lambda_i) < 0$ for $i = 1, 2, \dots, k$ $(k \leq h)$. Because S^* is an $(A+BKC)$-invariant that externally stabilizable then sub matrix (M'_{22}) in (7) implies $\mathrm{Re}(\lambda_i) < 0$ for $i = 1, 2, \dots, k$ $(k \leq (n - h))$. Based on (8) implies:

$\det (\lambda I - M') = \det (\lambda I - M) = \det (\lambda I_h - M'_{11}) \times \det (\lambda I_{n-h} - M'_{22})$

Therefore if $\mathrm{Re}(\lambda_i) < 0$ for $i = 1, 2, \dots, k$ $(k \leq h)$ and $\mathrm{Re}(\lambda_i) < 0$ for

$i = 1, 2, \dots, k$ $(k \leq (n - h))$ implies $\mathrm{Re}(\lambda_i) < 0$ for $i = 1, 2, \dots, k$ $(k \leq n)$

Therefore based on Definition 2.8, $\sigma(A + BKC) \subset C_g$.

Because Im $(E) \subset S^* \subset Ker(\mathrm{D})$, $(A+BKC)S^* \subset S^*$ and $\sigma(A+BKC) \subset C_g$, then disturbance rejection problem at (A, B, C, D, E) system with the system class is $\gamma^* \subset Ker(D) + S_*$ with stability by static output feedback can be solved. $\square$

4. Conclusion

Disturbance rejection problem at (A, B, C, D, E) system with the system class is $\gamma^* \subset Ker(D) + S_*$ with stability by static output feedback can be solved if and only if S^* is an (A, B)-invariant subspace that internally stabilizable and externally stabilizable.

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