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A Study of Energy on $\boldsymbol{k}$ −Splitting and $\boldsymbol{k}$ −Shadow Graphs

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Abstract

Let $G = (V, E)$ be a graph. The k −splitting graph of G, denoted by $S_k(G)$, is a graph constructed by adding to each vertex v of G as many as k new vertices such that the k new vertices are adjacent to the vertices that are adjacent to v. The k −shadow graph of G, denoted by $D_k(G)$, is a graph constructed by taking k copies of G and connecting each of these vertices with each of its neighboring vertices. The energy of a graph G is defined as the sum of the absolute values of all eigenvalues in the matrix for the graph. In this article, we study the energy of the k −splitting graph and the k −shadow graph, which are the energy of the adjacency matrix, the energy of the maximum degree matrix, the energy of the minimum degree matrix. We also revise the Sombor energy of the k −splitting graph and the k −shadow graph and we compare this result with the results carried out by previous researchers.

Keywords: k −shadow graph, k −splitting graph, maximum degree energy, minimum degree energy, Sombor energy

1 Introduction

A graph $G = (V, E)$ is a system consisting of a finite non-empty set of vertices, denoted by $V(G)$, and a finite set of edges, denoted by $E(G)$, with $E(G) \subseteq V(G) \times V(G)$. The order and size of G are defined as the number of its vertices and its edges, respectively. The application of graph theory can be seen in [1, 2, 3, 4, 5, 6, 7]. In this article, graph G is referred to simple connected graphs that is an undirected graph which has no loops and multiple edges. We refer to the basic material related to graph theory on [8]. In 2017, Vaidya and Popat introduced new special types of graphs, namely the k-splitting graph and the k-shadow graph [9]. The k −splitting graph of G, denoted $S_k(G)$, is the graph obtained by adding to each vertex $v \in V(G)$ as many as k new vertices $u_1, \dots, u_k$ such that each vertex u_i is adjacent to every vertex that is adjacent to v, where $i = 1,2, \dots, k$. The k −shadow graph of G, denoted $D_k(G)$, is a graph obtained by taking k copies of G, say $G_1, \dots, G_k$, then connecting every vertex u in G_i to the neighbors of the corresponding vertex v in G_j.

The adjacency matrix of a graph G of order n, denoted by $A(G)$, is an $n \times n$ matrix with the (i, j) −th entry is 1 if vertex u_i is adjacent to vertex u_j, and 0 otherwise. The maximum degree matrix of G, denoted $M(G)$, is an $n \times n$ matrix with the $(i,j)-$th entry defined as the maximum degree of a vertex u_i and u_j if u_i and u_j are adjacent, and 0 otherwise [10]. The minimum degree matrix of G, denoted by $m(G)$, is an $n \times n$ matrix with the $(i,j)-$th entry is defined as the minimum degree of a vertex u_i and u_j if u_i and u_j are adjacent, and 0 otherwise [11]. The Sombor matrix of G, denoted by $S(G)$, is an $n \times n$ matrix with the $(i,j)-$th entry is $\sqrt{\left(d(u_i)\right)^2 + \left(d(u_j)\right)^2}$ if u_i and u_j are adjacent, and 0 otherwise [12]. Some examples of uses of matrices on graphs can be found in [13, 14, 15].

The concept of energy in graphs was first introduced by Gutman in 1978 who was inspired by Huckel's molecular orbital approach [16]. The energy of a graph G, referred to as the energy of the adjacency matrix of G and denoted by $En(G)$, is defined as the sum of the absolute values of the eigenvalues of $A(G)$. The maximum degree energy of G, denoted by $EM(G)$, is defined as the sum of the absolute values of the eigenvalues of $M(G)$. The minimum degree energy of G, denoted by $Em(G)$, is defined as the sum of the absolute values of the eigenvalues of $m(G)$. The Sombor energy of G, denoted by $ES(G)$, is defined as the sum of the absolute values of the eigenvalues of $S(G)$.

Many researchers have discovered the energy of the $k-$splitting graphs and the $k-$shadow graphs. In 2017, Vaidya and Popat discovered the energy of the $k-$splitting graphs and the $k-$shadow graphs [9]. In 2019, Chu et al. discovered the maximum degree energy and minimum degree energy of splitting graphs [17]. In 2022, Rao et al. discovered the maximum degree energy and minimum degree energy of $k-$splitting graphs and corrected Chu et al.'s results [18]. Rao et al. also discovered the maximum degree energy and minimum degree energy of $k-$shadow graphs. In 2022, Singh and Patekar discovered the Sombor energy of $k-$splitting and $k-$shadow graphs [19]. Motivated by [9, 18, 19], in this article we review the various energies of $k-$splitting graphs and $k-$shadow graphs. The study includes energy, maximum degree energy, and minimum degree energy. We also revise the Sombor energy of the $k-$splitting graph and the $k-$shadow graph and we compare it with the results in [19].

The discussion in this article is divided into 4 sections. In the second section, we review some energy of the $k-$splitting graphs and the $k-$shadow graphs, which are energy, maximum degree energy, and minimum degree energy. In the third section, we give the Sombor energy of $k-$splitting graph and $k-$shadow graph and compare it with previous results. The last section is the conclusion of this article.

2 Preliminaries

In this section, we review some energy of the $k-$splitting graphs and the $k-$shadow graphs that have been discovered by [9, 18, 19]. We first defining the Kronecker product matrix and the important properties of the eigenvalues of the Kronecker product matrix. Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$. The Kronecker product matrix of A and B, denoted by $A \otimes B$, is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{bmatrix} = [a_{ij}B]_{np\times mq}. \quad (1)$$

We will use this proposition in the next section to prove our theorems.

Proposition 1. [20] *Let* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{m\times m}$. *If* ξ *and* γ *is eigenvalue of* A *and* B, *respectively, with corresponding eigen vector* **x** *and* **y**, *then* $\xi\gamma$ *eigenvalue of* $A \otimes B$ *with vector eigen* **x** $\otimes$ **y.**

Next, we give an example of the $k-$splitting graphs and the $k-$shadow graphs that are constructed from the C_4 graph and the P_4 graph.

Example 2. Consider a cyclic graph of order 4, namely C_4. From the C_4 graph, the graph $S_2(C_4)$ and $D_3(C_4)$ are constructed as shown below.

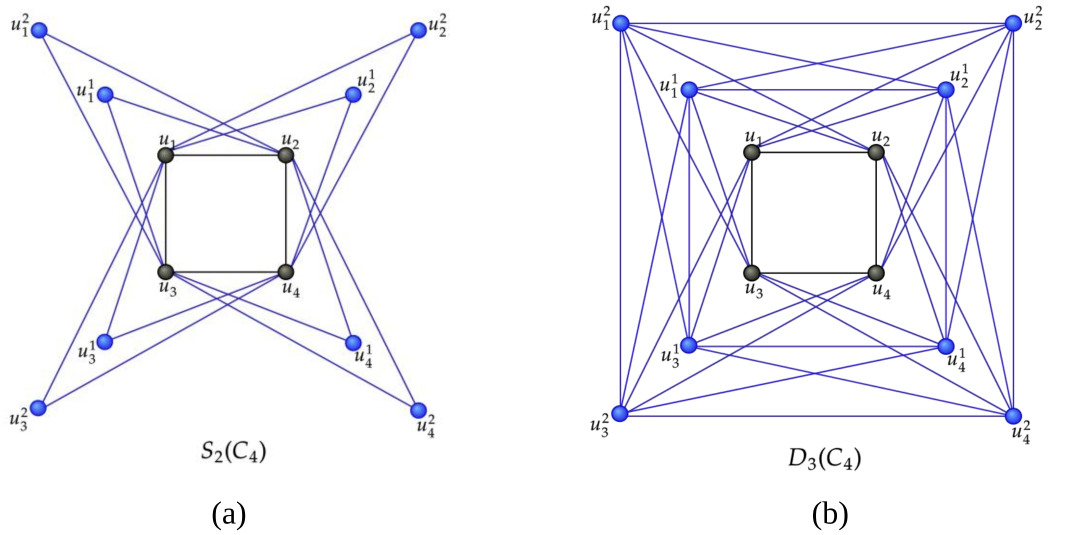


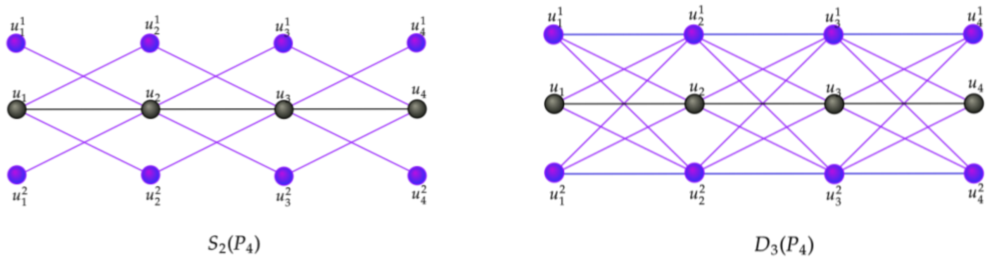
Figure 1. Graph (a). $S_2(C_4)$ and (b). $D_3(C_4)$

The construction of $S_2(C_4)$ and $D_3(C_4)$ are explained below.

- To construct a $S_2(C_4)$ graph, for each vertex $u_i, i = 1, \dots, 4$, add two new vertices, namely u_i^1 and u_i^2. Since u_1 is adjacent to u_2 and u_3, then u_1^1 and u_1^2 are both adjacent to u_2 and u_3 and so on for each vertex.

- To construct a $D_3(C_4)$ graph, take 3 copies of C_4 graph such that each vertex $u_i, i = 1, \ldots, 4$ is corresponds to u_i^1 and u_i^2. Since u_1 is adjacent to u_2 and u_3, then u_1^1 and u_1^2 are both adjacent to $u_2, u_2^1, u_2^2, u_3, u_3^1, u_3^2$, and so on for each vertex.

Example 3. Consider a path graph of order 4, namely P_4. From the path graph P_4, a graph $S_2(P_4)$ and $D_3(P_4)$ is constructed as shown below.



(a) (b)

Figure 2. Graph (a). $S_2(P_4)$ and (b). $D_3(P_4)$

The difference between k −splitting graphs and k −shadow graphs is in it's order and size. In a k −splitting graph, the order of $S_k(G)$ is $(k+1)|V(G)|$, while in the k −shadow graph, the the order of $D_k(G)$ is $k|V(G)|$. In a k −splitting graphs, the size of $S_k(G)$ is $(2k+1)|E(G)|$, while in k −shadow graphs, the size of $D_k(G)$ is $k^2|E(G)|$.

In the following, we give the energy of k −splitting graphs and k −shadow graphs that are discovered by Vaidya and Popat [9].

Theorem 4. [9] *If G is simple connected graph, then $En(S_k(G)) = En(G)\sqrt{1+4k}$.*

Theorem 5. [9] *If G is simple connected graph, then $En(D_k(G)) = k \cdot En(G)$.*

The following example is related to Theorem 4 and Theorem 5.

Example 6. Consider a $S_2(P_4)$ graph and $D_3(P_4)$ graph in Example 3. By direct calculation we have $En(S_2(P_4)) = 6\sqrt{5}$ and $En(D_3(P_4)) = 6\sqrt{5}$. In other way, it is clear that $En(P_4) = 2\sqrt{5}$ so

$$En(S_2(P_4)) = 6\sqrt{5} = 3 \cdot 2\sqrt{5} = \sqrt{4 \cdot 2 + 1} \cdot En(P_4),$$

as stated in Theorem 4. Also,

$$En(D_3(P_4)) = 6\sqrt{5} = 3 \cdot 2\sqrt{5} = 3 \cdot En(P_4),$$

as stated in Theorem 5.

The following theorem is the maximum degree energy of a k −splitting graph constructed from a class of regular graphs [18].

Theorem 7. [18] *If G is r −regular graph, then $EM(S_k(G)) = (k+1)\sqrt{4k+1}EM(G)$.*

The following example is related to Theorem 7.

Example 8. Consider a $S_2(C_4)$ graph in Example 2. By direct calculation we have $EM\big(S_2(C_4)\big) = 72$. In other way, it is clear that $EM(C_4) = 8$. Note that

$$EM\big(S_2(C_4)\big) = 72 = 3 \cdot 3 \cdot 8 = (2+1) \cdot \sqrt{1 + 4 \cdot 2} \cdot EM(C_4),$$

as stated in Theorem 7.

The maximum degree energy of a k −shadow graph is given as follows.

Theorem 9.[18] *For every graph G holds $EM\big(D_k(G)\big) = k^2 EM(G)$.*

The following example is related to Theorem 9.

Example 10. Consider a $D_3(P_4)$ graph in Example 3. By direct calculation we have $EM\big(D_3(P_4)\big) = 18\sqrt{5}$. In other way, it is clear that $EM(P_4) = 2\sqrt{5}$. Note that

$$EM\big(D_3(P_4)\big) = 18\sqrt{5} = 9 \cdot 2\sqrt{5} = 3^2 \cdot EM(P_4),$$

as stated in Theorem 9.

The following theorem is the minimum degree energy of a k −splitting graph constructed from a class of regular graphs.

Theorem 11.[18] *If G is r −regular graph, then $Em\big(S_k(G)\big) = \sqrt{4k + (k+1)^2}Em(G)$.*

The following example is related to Theorem 11.

Example 12. Consider a $S_2(C_4)$ graph in Example 2. By direct calculation we have $Em\big(S_2(C_4)\big) = 8\sqrt{17}$. In other way, it is clear that $Em(C_4) = 8$. Note that

$$Em\big(S_2(C_4)\big) = 8\sqrt{17} = Em(C_4)\sqrt{4 \cdot 2 + (2+1)^2},$$

as stated in Theorem 11.

Next, the minimum degree energy formula of the k −shadow graph is given as follow.

Theorem 13. [18] *For every graph G holds $Em\big(D_k(G)\big) = k^2 Em(G)$.*

The following example is related to Theorem 13.

Example 14. Consider a $D_3(P_4)$ graph in Example 3. By direct calculation we have $Em\big(D_3(P_4)\big) = 36\sqrt{2}$. In other way, it is clear that $Em(P_4) = 4\sqrt{2}$. Note that

$$Em\big(D_3(P_4)\big) = 36\sqrt{2} = 9 \cdot 4\sqrt{2} = 3^2 \cdot Em(P_4),$$

as stated in Theorem 13.

Singh and Patekar gave the Sombor energy of k −splitting graphs and k −shadow graphs as follow.

Theorem 15. [19] *If G is r −regular graph, then $ES\big(S_k(G)\big) = r(k+1)\sqrt{2}En(G)$.*

Theorem 16. [19] *For every graph G holds $ES\big(D_k(G)\big) = rk\sqrt{2 + 8k}En(G)$.*

3 Result and Discussion

In this section, we give the Sombor energy on $k-$splitting graphs and k-shadow graphs. We first give a counter example to Theorem 15 and Theorem 16.

Example 17. Consider a $S_2(C_4)$ graph in Example 2. By direct calculation we have $ES(S_2(C_4)) = 56\sqrt{2}$. In other way, it is clear that $En(C_4) = 4$. Note that

$$ES(S_2(C_4)) = 56\sqrt{2} \neq 24\sqrt{2} = 2 \cdot (2+1) \cdot (\sqrt{2}) \cdot 4,$$

which does not satisfy Theorem 14. Also

$$ES(D_3(C_4)) = 72\sqrt{2} \neq 24\sqrt{26} = 2 \cdot 3 \cdot \left(\sqrt{2+8\cdot 3}\right) \cdot 4,$$

which does not satisfy Theorem 15.

We found inaccuracy in Theorem 14 and Theorem 16. The correction of Sombor energy for $k-$splitting graph constructed from regular graph is as follows.

Theorem 18. *If G is $r-$regular graph, then $ES(S_k(G)) = r\sqrt{2(2k+1)(k+1)^2 + 4k}En(G)$.*

Proof. Let G be a $r-$regular graph of order n. The Sombor matrix of $S_k(G)$ is given by

$$S(S_k(G)) = \begin{bmatrix} S_1 & S_2 & \dots & S_2 \\ S_2 & O & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ S_2 & O & \dots & O \end{bmatrix}_{n(k+1)},$$

where S_1 and S_2 are $n \times n$ matrices with entry of S_1 is defined by

$$a_{ij} = \begin{cases} (k+1)r\sqrt{2}, & \text{if } u_i \text{ and } u_j \text{ adjacent} \\ 0, & \text{otherwise} \end{cases},$$

and entry of S_2 is defined by

$$b_{ij} = \begin{cases} r\sqrt{(k+1)^2+1}, & \text{if } u_i \text{ and } u_j \text{ adjacent} \\ 0, & \text{otherwise} \end{cases}.$$

Note that $S_1 = (k+1)r\sqrt{2}A(G)$ and $S_2 = r\sqrt{(k+1)^2+1}A(G)$, so matrix $S(S_k(G))$ can be written as

$$S(S_k(G)) = B \otimes A(G),$$

with

$$B = \begin{bmatrix} (k+1)r\sqrt{2} & r\sqrt{(k+1)^2+1} & \dots & r\sqrt{(k+1)^2+1} \\ r\sqrt{(k+1)^2+1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r\sqrt{(k+1)^2+1} & 0 & \dots & 0 \end{bmatrix}_{k+1}.$$

Let μ be eigenvalues of B. The characteristic equation of B is

$$\mu^{k-1}(\mu^2 - (k+1)r\sqrt{2} - r^2k(k+1)^2 + 1)) = 0.$$

The eigenvalues of B are $\mu_{1,2} = \frac{r(k+1)\sqrt{2}\pm\sqrt{4k^3+10k^2+12k+2}}{2}$, and $\mu_3 = 0$ with multiplicity $k-1$. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of matrix $A(G)$. By Proposition 1, the Sombor energy of graph $S_k(G)$ is

$$ES\big(S_k(G)\big) = r\sqrt{4k^3+10k^2+12k+2}\sum|\lambda_i| = r\sqrt{2(2k+1)(k+1)^2+4k}En(G).$$

■

Example 19. Consider a $S_2(C_4)$ graph in Example 2. By direct calculation we have $ES\big(S_2(C_4)\big) = 56\sqrt{2}$. In other way, it is clear that $En(C_4) = 4$. Note that

$$ES\big(S_2(C_4)\big) = 56\sqrt{2} = 14\sqrt{2}\cdot 4 = 2\cdot\sqrt{2(2\cdot 2+1)(2+1)^2+4\cdot 2}\cdot En(C_4),$$

as stated in Theorem 18.

Next, we give the correction of Sombor energy for k −shadow graph constructed from regular graph is as follows.

Theorem 20. *If G is r −regular graph, then* $ES\big(D_k(G)\big) = k^2 r\sqrt{2}\, En(G)$.

Proof. Let G be a r −regular graph of order n. Since the k −shadow graph of r −regular graph is kr −regular graph, the Sombor matrix of graph $D_k(G)$ is given by

$$S\big(D_k(G)\big) = \begin{bmatrix} S_3 & S_3 & \dots & S_3 \\ S_3 & S_3 & \dots & S_3 \\ \vdots & \vdots & \ddots & \vdots \\ S_3 & S_3 & \dots & S_3 \end{bmatrix}_{nk},$$

where S_3 is $n\times n$ with entry defined as

$$c_{ij} = \begin{cases} kr\sqrt{2}, & \text{if } u_i \text{ and } u_j \text{ adjacent} \\ 0, & \text{otherwise} \end{cases}.$$

Note that $S_3 = kr\sqrt{2}A(G)$, so matrix $S\big(D_k(G)\big)$ can be written as

$$S\big(D_k(G)\big) = \begin{bmatrix} kr\sqrt{2} & kr\sqrt{2} & \dots & kr\sqrt{2} \\ kr\sqrt{2} & kr\sqrt{2} & \dots & kr\sqrt{2} \\ \vdots & \vdots & \ddots & \vdots \\ kr\sqrt{2} & kr\sqrt{2} & \dots & kr\sqrt{2} \end{bmatrix}_k \otimes A(G) = \Big(\big(kr\sqrt{2}\big)J_k\Big)\otimes A(G),$$

where J_k is $k\times k$ matrix with entries all 1. It is clear that the eigenvalues of J_k are k and 0 with multiplicity $k-1$. Thus, eigenvalues of matrix $\big(kr\sqrt{2}\big)J_k$ are $k^2 r\sqrt{2}$ and 0 with multiplicity $k-1$. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of $A(G)$. By Proposition 1 and in similar way, the Sombor energy of graph $D_k(G)$ is

$$ES\big(D_k(G)\big) = \sum_{i=1}^{n}\big|k^2 r\sqrt{2}\big||\lambda_i| = k^2 r\sqrt{2}A(G).$$

■

Example 21. Consider a $S_2(C_4)$ graph in Example 2. By direct calculation we have $ES\big(D_3(C_4)\big) = 72\sqrt{2}$. In other way, it is clear that $En(C_4) = 4$. Note that

$$ES\big(D_3(C_4)\big) = 72\sqrt{2} = 9 \cdot 2 \cdot \sqrt{2} \cdot 4 = 3^2 \cdot 2 \cdot \sqrt{2} \cdot En(C_4),$$

as stated in Theorem 20.

4 Conclusion

Based on the results and discussion above, we obtain the Sombor energy of the $k-$splitting graphs and $k-$shadow graphs for $r-$regular graphs are $ES\big(S_k(G)\big) = r\sqrt{2(2k+1)(k+1)^2 + 4k}En(G)$ and $ES\big(D_k(G)\big) = k^2 r\sqrt{2}\, En(G)$, respectively. For the next researches, the researchers can investigate:

1. the maximum degree energy and minimum degree energy of the $k-$splitting graphs for any graph G,
2. the Sombor energy of the $k-$splitting graphs and the $k-$shadow graphs for any graph G, and
3. other energies of $k-$shadow graphs and $k-$splitting graphs.

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