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HIGH ORDER B-SPLINE COLLOCATION METHOD AND ITS APPLICATION FOR HEAT TRANSFER PROBLEMS

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Abstract – High order B-spline collocation for solving boundary value problem is presented in this paper. The approach employs high order B-spline basis functions with high approximation and continuity properties to handle problem domain with scattered or random distribution of knot points. Using appropriate B-spline basis function construction, the new approach introduces no difficulties in imposing both Dirichlet and Neumann boundary conditions in the problem domain. Several numerical examples in arbitrary domains, both regular and irregular shaped domains, are considered in the present study. In addition, simulation results concerning with heat transfer applications are further presented and discussed.

Keywords: B-splines; high order; approximation; meshless; arbitrary domains; heat transfer.

1. Introduction

Numerical methods such as finite difference, finite element and boundary element have become more preference tool during the past decades in solving various engineering and science problems in particular those dealing with complicated shaped domains as well as moving interface problems involving care boundary conditions treatment (Bathe, 1996; Ohayon, 2004; Yu *et al*., 2010). The aforementioned methods are commonly considered as mesh-based methods.

In recent years, considerable efforts have been also devoted to introduce and develop a new class of numerical methods which do not rely on the information of element connectivity or predefined meshes, so-called mesh-less methods. Only a set of nodes is used to represent the whole problem domain and also no any priori information on the node relationships is required for the approximation of the unknown functions of the potential variables in the methods. It has been shown that the cumbersome task of mesh preparation commonly attributed to the mesh-based methods can be now largely eliminated resulting in more flexibility in handling arbitrary domain applications. Developments and successful applications of the mesh-less methods can be seen, for examples, in (Leitao *et al*., 2007; Belytschko *et al*., 1994; Atluri and Zhu, 1998; Zhang *et al*., 2000; Liu *et al*., 2005; Boroomand *et al*., 2009; Liu and Gu, 2003; Dehghan and Ghesmati, 2010) and references therein.

Recently, another class of numerical methods based upon B-spline basis function has been also gaining much attention due to the interesting characteristics of the basis function. It has been shown that B-spline basis function poses high approximation capabilities and continuity along with compact support and locality properties. Hence, there have been also increasing interest and motivation in applying B-spline based methods as viable numerical method in engineering (de Boor, 2001; Chui, 1988; Farin, 2002; Salomon, 2006; Hoggar, 2006; Hollig, 2003; Chaniotis and Poulikakos, 2004).

While the B-spline based methods have shown their promising applications for solving the boundary value problems, the applications, however, are particularly limited to regular domains or regular distribution of knot points, making their extensions to irregular domains rather complicated and not always straightforward (Burla and Kumar, 2008; Botella, 2002; Jator and Sinkala, 2007; Johnson, 2005). On the other hand, mesh-less methods offer great flexibilities in handling irregular nodes and shaped domains, thus very suitable for the analysis of boundary value problems in complex shaped domains as previously highlighted.

In the present paper, assessment of high order B-spline collocation for solving boundary value problem is presented. The approach presented employs high order B-spline basis functions with high approximation and continuity properties to handle problem domain with scattered or random distribution of knot points.

It is noted here the advantage of using B-spline is that one can choose any order of approximation when constructing the B-spline basis functions, thus it can be designed to meet with necessary requirement for the problems considered. Unlike Galerkin or collocation based meshless methods that sometimes need special treatment or additional term for imposing Dirichlet or Neumann boundary conditions (Wang and Qin, 2006; Liu and Tai, 2006), the imposition of both Dirichlet and Neumann boundary conditions in the new approach can be easily facilitated by choosing an appropriate B-spline basis function construction using the open-uniform knot vector. Using the knot vector, B-spline is efficiently designed to have Kronecker delta condition which is useful property for interpolation. It has been shown that the approach introduces no difficulties in imposing both Dirichlet and Neumann boundary conditions in the boundary problem domain.

Several numerical applications in arbitrary domains, both regular and irregular shaped domains, are considered in the present study and the simulation results concerning heat transfer applications are further presented and discussed.

2. High Order B-splines Approximation

B-spline curve is a piecewise polynomial function that is connected continuously by different curve segments, which is contained in the convex hull of its control poly-lines. B-spline has minimal support with respect to a given degree, smoothness and domain partition (de Boor, 2001).

Three important components define the B-spline curves, namely: knot vector, basis or blending functions and control points. Mathematically, the B-spline curve is defined along the parametric interval t as:

$$\mathbf{C}(t) = \sum_{i=1}^{n+1} \mathbf{C}_i \, N_{i,k}(t) \tag{1}$$

where: [$\mathbf{C}_i$: $i = 1, 2, \ldots, n+1$] are the control points, which are in general do not lie on the curve, k is the order of B-spline basis function, meaning that it represents the polynomial of order $p = k - 1$, $N_{i,k}(t)$ is the normalized B-spline basis function, which is described by the order k and a non-decreasing sequence of real numbers in the knot vector Ξ: [t_i: i = 1, . . . , $n+k+1$], satisfying the Schoenberg-Whitney condition and t is the parameterized coordinate.

The Cox-de Boor recursion formula is used to obtain B-spline basis functions of the intended order $N_{i,k}(t)$, starting from the first order B-spline function as follows:

$$N_{i,1}(t) = \begin{cases} 1, t_i \le t < t_{i+1} \\ 0, \text{otherwise} \end{cases} \tag{2}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t) \quad \text{for } 2 \le k \le n+1$$

where t_i are the elements of the knot vector previously mentioned. It is important to note that B-spline basis functions also have the partition of unity property.

The following relationship between the total number of knots in the knot vector m, the control points $n+1$ and the order k holds:

$$m = n + k + 1 \tag{3}$$

Eq. (3) describes that each control point needs basis function which is in the relationship with the knots in the knot vector chosen.

Furthermore, the d-th derivatives of the B-spline can be also easily computed as follows:

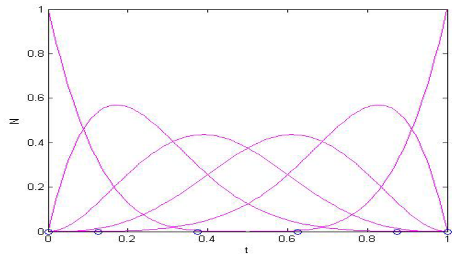
$$\mathbf{C}^{(d)}(t) = \sum_{i=1}^{n+1} \mathbf{C}_i \, N^{(d)}{}_{i,k}(t) \tag{4}$$

The application for higher dimensions, 2D or 3D is served by taking the tensor product of the one-dimensional B-splines. For 2D domain, the tensor product is defined as follows:

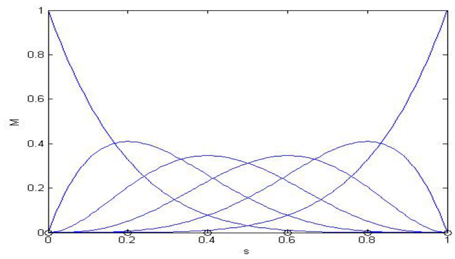
$$\mathbf{D}(t,s) = \sum_{i=1}^{n+1} \sum_{j=1}^{m+1} \mathbf{D}_{i,j} \, N_{i,k}(t) M_{j,l}(s) \ , \qquad t_{min} \le t \le t_{max},\ s_{min} \le s \le s_{max} \tag{5}$$

where: $N_{i,k}(t)$ and $M_{j,l}(s)$ are the B-spline basis functions of k-th and l-th orders along the parametric directions t and s, respectively. $\mathbf{D}_{i,j}$ is now the corresponding coefficients of the 2D approximation.

Fig. 1 depicts the B-spline basis functions resulted in from the knot vectors Ξ = [0,0,0,0,0,1/2,1,1,1,1,1] and Ξ = [0,0,0,0,0,0,1,1,1,1,1,1], respectively.



(a)



(b)

Fig. 1. (a) 5th B-spline basis functions of the knot vector Ξ = [0,0,0,0,0,1/2,1,1,1,1,1] and (b) 6th B-spline basis functions of the knot vector Ξ = [0,0,0,0,0,0,1,1,1,1,1,1].

Further, to deal with Dirichlet and Neumann boundary conditions in the boundary value problems considered, the knot vector with k-multiple knots (open-uniform knot vector) is chosen in the present study, particularly the one without mid-knot(s), as shown in Fig. 1(b). Using the knot vector, B-spline is efficiently designed to have Kronecker delta condition which is useful property for interpolation, thus makes it easy to impose the Dirichlet boundary conditions. The imposition of Neumann boundary conditions is also facilitated easily by the use of the knot vector.

3. Collocation Method for Boundary Value Problem

Consider the following general form of a boundary value problem:

$$\begin{aligned} &\mathbf{Lu} = \mathbf{f} && in \quad \Omega \\ &\mathbf{B}^{h}\mathbf{u} = \mathbf{h} && on \quad \partial\Omega^{h} \\ &\mathbf{B}^{g}\mathbf{u} = \mathbf{g} && on \quad \partial\Omega^{g} \end{aligned} \tag{6}$$

where **Ω** is the problem domain, $\partial\Omega^h$ is the Dirichlet boundary, $\partial\Omega^g$ is the Neumann boundary and piecewise smooth boundary $\partial\Omega = \partial\Omega^h \cup \partial\Omega^g$.

For Poisson equation, the following notations are introduced: **L** is the Laplace operator, ∇^2, $\mathbf{B}^h = 1$, $\mathbf{B}^g = (\partial / \partial n)$, **n** is the vector of normal directions, whose components n_i, to the Neumann boundary. Furthermore, $\mathbf{f} = f(\mathbf{x},\mathbf{u})$ is the potential function, where $\mathbf{x} \in R^d$ represents the vector of position and d is the dimension of the problem domain **Ω**, **h** and **g** are the prescribed values of the Dirichlet and Neumann BCs.

In collocation method, the residuals are enforced to be zeros at a set of collocation points: *Np* is a set of collocation points in Ω, *Nb* is a set of collocation points in $\partial\Omega^h$ and *Nq* is a set of collocation points in $\partial\Omega^g$, thus forming a set of discrete equations defining the boundary value problem.

4. Results and Discussion

Several numerical examples are presented in this section to show the capability of the present approach for solving boundary value problems in both regular and irregular domains. To assess the numerical performance of the approach, this error norm is used:

$$E_n = \sqrt{\frac{\sum_{i=1}^{n}\left[u(x_i) - \bar{u}(x_i)\right]^2}{\sum_{i=1}^{n} u(x_i)^2}} \tag{7}$$

n is the number of collocation points, $u(x_i)$ and $\bar{u}(x_i)$ represent the exact solution and the present method approximation, respectively.

4.1. *Poisson equation in complex domain-I*

The following Poisson equation:

$$\nabla^2 u = -\frac{751\pi^2}{144}\sin\left(\frac{\pi x}{6}\right)\sin\left(\frac{7\pi x}{4}\right)\sin\left(\frac{3\pi y}{4}\right)\sin\left(\frac{5\pi y}{4}\right) + \frac{7\pi^2}{12}\cos\left(\frac{\pi x}{6}\right)\cos\left(\frac{7\pi x}{4}\right)\sin\left(\frac{3\pi y}{4}\right)\sin\left(\frac{5\pi y}{4}\right)$$
$$+\frac{15\pi^2}{8}\sin\left(\frac{\pi x}{6}\right)\sin\left(\frac{7\pi x}{4}\right)\cos\left(\frac{3\pi y}{4}\right)\cos\left(\frac{5\pi y}{4}\right) \tag{8}$$

is considered and applied on the complex region-I as shown in Fig. 2.

The exact solution of the Poisson problem is given by:

$$u = \sin\left(\frac{\pi x}{6}\right)\sin\left(\frac{7\pi x}{4}\right)\sin\left(\frac{3\pi y}{4}\right)\sin\left(\frac{5\pi y}{4}\right) \tag{9}$$

and further depicted in Fig. 3.

The accuracy of the high order B-spline collocation method obtained was 3.237E-5 using 534 collocation points.

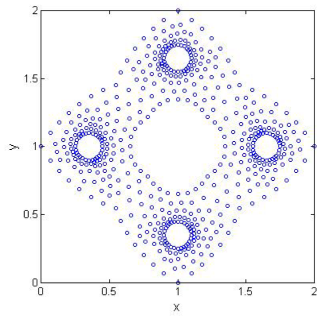


Fig. 2. Complex domain-I with irregular knot points distribution.

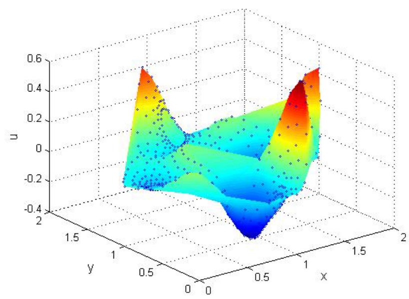


Fig. 3. The exact solution of the Poisson problem over the complex domain-I (blue circle points represent the approximation results evaluated at the collocation points).

4.2. *Poisson equation in complex domain-II*

Given the Poisson equation:

$$\nabla^2 u = -18\pi^2 \sin(3\pi x)\sin(3\pi y) \tag{10}$$

applied on the complex region-II as in Fig. 4. The exact solution of the Poisson problem is given as:

$$u = \sin(3\pi x)\sin(3\pi y) \tag{11}$$

and further depicted in Fig. 4. In addition, the accuracy of the high order B-spline collocation method in solving for the Poisson problem over the complex domain-II is shown in Table 2.

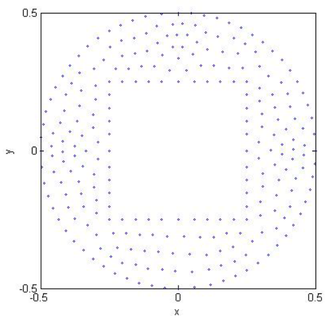


Fig. 4. Complex domain-II with 260 knot points distribution.

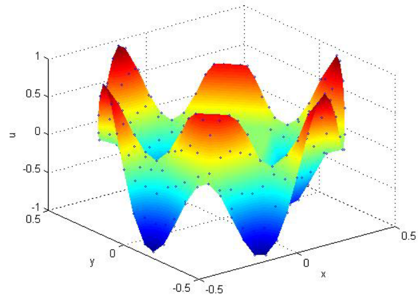


Fig. 5. The exact solution of the Poisson problem over the complex domain-II (blue circle points represent the approximation results evaluated at the collocation points).

Table 1. Accuracy of the high order B-spline collocation method for the Poisson problem over the complex domain-II

Problem domain	Number of collocation points	Error (E_n)
Complex domain-II	96	2.8E-3
	260	4.826E-7
	526	3.293E-7
	668	4.799E-7

4.3. ***Heat transfer problem-I***

Consider the following heat transfer problem defined in Ω = [0, 1] x [0, 1] with Dirichlet and Neumann boundary conditions and discretized with 488 collocation points as shown in Fig. 6. The simulation result of distribution temperature for the heat transfer problem involving Dirichlet and Neumann boundary conditions is depicted in Fig. 7 and further compared with analytical solution and benchmark solutions from several other numerical methods (Zahiri *et al*., 2009), as presented in Table 2.

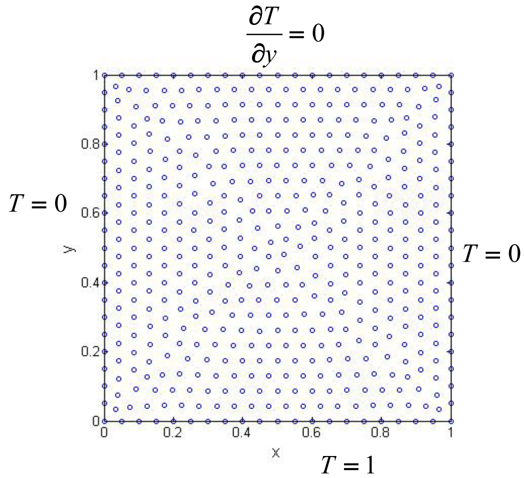


Fig. 6. Heat transfer problem-I with the corresponding boundary conditions.

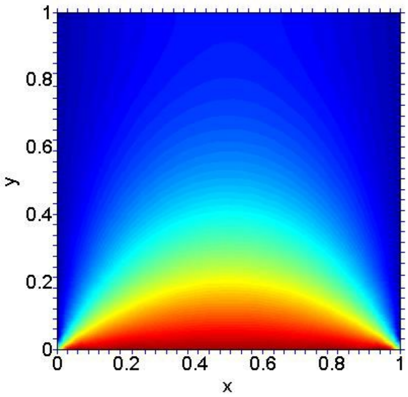


Fig. 7. Distribution of temperature in square domain for heat transfer problem-I.

From Table 2, it can be observed that the present approach produced high approximation in solving the heat transfer problem compared to the previously mentioned methods. It is also worth to note that the present approach utilized less collocation points that those employed by the other numerical methods showing the effectiveness and efficiency of the high order B-spline collocation method.

Table 2. Comparison of temperature profile T at $x = 0.5$ for heat transfer problem in square domain

Position	**Solutions & Number of Collocation Points**			
y	Analytical	Present	LRPIM	MWS
	N/A	488	1156	1156
0	1	1	1	1
0.1	0.8047	0.8047	0.8041	0.8035
0.2	0.6271	0.6272	0.6262	0.6254
0.3	0.4782	0.4782	0.4771	0.4762
0.4	0.3606	0.3607	0.3598	0.3589
0.5	0.2718	0.2719	0.2712	0.2705
0.6	0.2071	0.2071	0.2066	0.2059
0.7	0.1617	0.1617	0.1614	0.1608
0.8	0.1320	0.1320	0.1318	0.1314
0.9	0.1152	0.1151	0.1151	0.1147
1	0.1098	0.1097	0.1097	0.1093

4.4. ***Heat transfer problem-II***

The following heat transfer problem which is defined in L-shaped domain is considered. The problem poses sharp corner representing singularity in the flux solution [26]. The domain is discretized with 2193 collocation points as depicted in Fig. 8.

The simulation results obtained are further compared with both analytical and BEM solutions (Ramsak and Skerget, 2004) as shown in Table 3. Furthermore, the temperature distribution in the L-shaped domain is depicted in Fig. 9.

It is observed that the present approach can capture well the singularity nature of the problem and the results obtained by the approach are comparable with those obtained by BEM solutions. Moreover, it may be expected that for the problem involving singularity on the boundaries, the BEM techniques will be providing better results.

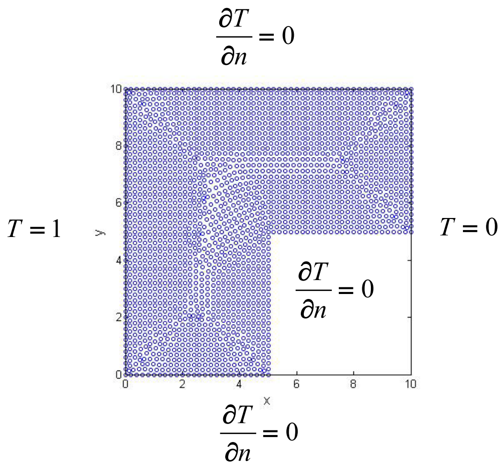


Fig. 8. Heat transfer problem-II in L-shaped domain.

Table 3. Comparison of temperature profile T at x = 0.5 for heat transfer problem in L-shaped domain

Position	**Solutions & Number of Collocation Points**			
y	Analytical	Present	HBEM	SBEM
	N/A	2193	N/A	N/A
0	0.906	0.816	0.910	0.893
2.5	0.868	0.793	0.881	0.857
5	0.667	0.654	0.731	0.667
7.5	0.566	0.585	0.523	0.572
10	0.547	0.574	0.553	0.553

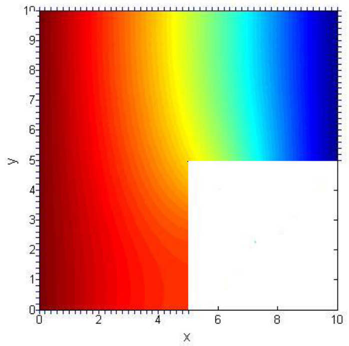


Figure 9 – Distribution of temperature in L-shaped domain for heat transfer problem-II.

4.5. *Approximation order and convergence studies*

There is great flexibility when working with B-spline in particular with respect to its approximation order as provided by the Cox-de Boor recursion formula, (2) and (3). From the numerical examples considered in the present study, it was found that the B-spline approximation of ≥ 10th is suitable to use giving a global handle in the approximation over the problem domain. It is noted here B-spline basis functions with the order of 15 and 17 were respectively employed for the Poisson problems in complex domain-I and II, while that of 17 and 13 were employed for the heat transfer problems I and II, respectively. It is important to note, however, that the use of B-spline with much higher order could introduce further computational cost. It resulted in un-symmetric and dense solution matrix. Working with such a matrix may lead to ill-conditioning problem relating to stability and robustness of an approximation method, particularly as the number of nodes increases. It may be very useful to use some pre-conditioning techniques or other techniques from wavelets for further improvement in the solution accuracy and efficiency. Sub-domain techniques may be also employed. The sub-domain technique has been implemented in BEM formulation and implementation, such as in (Ramsak and Skerget, 2004).

Lastly, it is also interesting to observe the convergence rate of the present global collocation approach. Fig. 10 depicts the convergence rate of the present approach for the Poisson problem-II. It is clear from Fig. 10 that the present approach has high convergence rate for the Poisson problem considered in the present study.

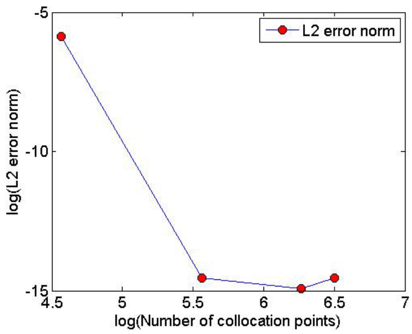


Fig. 10. Convergence rate of the present method for the Poisson problem-II.

5. Conclusions

In this paper, high order B-spline collocation method has been presented for solving boundary value problems. The method requires no mesh, thus eliminates the mesh burden in computation. Numerical results show applicability and capability of the proposed method for solving boundary value problems. Further implementations in 3D or other applications are interesting as subjects of further study.

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