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A WEIGHTED LEAST SQUARES B-SPLINE COLLOCATION METHOD FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract – A new, free-integration approach based upon B-spline for solving boundary value problem is introduced and presented in the paper, called weighted least squares B-spline collocation method. It combines high order B-spline basis functions with high approximation and continuity properties and weighted least squares method which is robust to deal with scattered or randomly knot points distribution. In addition, using appropriate designed B-spline basis function construction, the new approach introduces no difficulties in imposing both Dirichlet and Neumann boundary conditions in the problem domain. As a result, the effectiveness of the new approach is greatly enhanced with the flexibility to cope with both regular and irregular shaped domains. Numerical examples show the applicability and capability of the new approach for solving elliptic partial differential equations in arbitrary domains.

Keywords: Weighted B-splines; least square; approximation; meshless; arbitrary domains; elliptic PDEs.

1. Introduction

During the past 40 years, mesh-based methods such as finite difference, boundary element and in particular finite element method have becoming dominant numerical method in engineering for solving various potential problems in several important applications from elasticity to fluid-structure interaction (Bathe, 1996; Ohayon, 2004; Yu *et al*., 2010). Nevertheless, the use of mesh-based methods could introduce numerical difficulties in particular for problems involving large deformation or moving boundary problems where grid adaptivity or remeshing is commonly encountered. As a consequence, new meshes and projections of solution to the new meshes need to establish and perform frequently, which are often troublesome and time-consuming to address. In turn, this would degrade the quality of numerical solution pursued.

In recent years, considerable efforts have been devoted to introduce and develop a new class of numerical methods which do not rely on the information of element connectivity or predefined meshes, so-called meshless methods. Because only a set of nodes is used to represent the whole problem domain and boundaries and also no any priori information on the node relationships is required for the approximation of the unknown functions of the potential variables, the mesh-based methods related troubles and difficulties can be efficiently addressed. Developments and successful applications of meshless methods can be seen, for examples, in (Leitao *et al*., 2007; Belytschko *et al*., 1994; Atluri and Zhu, 1998; Zhang *et al*., 2000; Liu *et al*., 2005; Boroomand *et al*., 2009; Liu and Gu, 2003; Dehghan and Ghesmati, 2010) and references therein.

In addition, another class of numerical methods based upon B-spline basis function has been also gaining much attention in recent years due to the interesting characteristics of the basis function. It has been shown that B-spline basis function poses high approximation capabilities and continuity along with compact support and locality properties. Hence, there have been increasingly interest and motivation in applying B-spline based methods as viable numerical method in engineering (de Boor, 2001; Chui, 1988; Farin, 2002; Salomon, 2006; Hoggar, 2006; Hollig, 2003; Chaniotis and Poulikakos, 2004).

While the B-spline based methods have shown their promising applications for solving the boundary value problems, the applications, however, are particularly limited to regular domains or regular arrangement of knot points, making their extensions to irregular domains rather complicated and not always straightforward to handle (Burla and Kumar, 2008; Botella, 2002; Jator and Sinkala, 2007; Johnson, 2005). On the other hand, meshless methods offer great flexibilities in handling irregular shaped domains or randomly distributed nodes, thus very suitable for the analysis of boundary value problems in complex shaped domains (Belytschko *et al*., 1994; Atluri and Zhu, 1998; Zhang *et al*., 2000).

In the present paper, a new, free-integration approach based upon B-spline for solving boundary value problem is introduced, called weighted least squares B-spline collocation. It combines high order B-spline basis functions with high approximation and continuity properties and the weighted least squares method which is robust to deal with scattered or randomly points distribution. The advantage of using B-spline is that one can choose any order of approximation when constructing the B-spline basis functions, thus it can be designed to meet with necessary requirement for the problems considered. Furthermore, the utilization of weighted least squares formulation stabilizes the approximation of the method and further produces smooth solution in particular with respect to the randomly knot points distribution. Least squares method has been well-known as robust approach for function approximation of scattered and noisy data applied in various applications such as statistics, computer graphics, optimization, neural networks and meshless methods, among others (Wolberg, 2006; Haykin, 2009; Nocedal and Wright, 2006).

Moreover, unlike Galerkin or collocation based meshless methods that sometimes need special treatment or additional term for imposing Dirichlet or Neumann boundary conditions (Wang and Qin, 2006; Liu and Tai, 2006), the imposition of both Dirichlet and Neumann boundary conditions in the weighted least squares B-spline collocation method can be easily facilitated by choosing an appropriate B-spline basis function construction using the open-uniform knot vector. Using the knot vector, B-spline is efficiently designed to have Kronecker delta condition which is useful property for interpolation. As a result, the effectiveness of the new approach approximation is greatly enhanced with the flexibility to cope with both regular and irregular shaped domains. Numerical examples show the applicability and capability of the new approach for solving elliptic partial differential equations in arbitrary domains.

2. Weighted Least Squares (WLS) Approximation Using B-Spline Basis Functions

Consider the following functional, the square of the residual norm **e**:

$$\mathbf{\Pi} = \frac{1}{2}\|\mathbf{e}\| = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e} \qquad (1)$$

Consider further the WLS approximation of $u(\mathbf{x})$ defined at x which is stated as:

$$u(\mathbf{x}) \cong u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x})\, a_j(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}\mathbf{a} \qquad (2)$$

where: $p^{\mathrm{T}}(\mathbf{x})$ is the basis function of the special coordinates which is chosen as B-spline basis functions to be described later, m is the number of basis functions and a($\mathbf{x}$) is a vector of coefficients to be determined.

The residual **e** is now defined as:

$$\mathbf{e} = u^h(\mathbf{x}) - u(\mathbf{x}) \qquad (3)$$

Substituting (3) into (1) following by minimizing **Π** with respect to **a** and introducing further a weight matrix w($\mathbf{x}$) will result in:

$$\mathbf{wp}^{\mathrm{T}}\mathbf{p}\,\mathbf{a} = \mathbf{wp}^{\mathrm{T}}\mathbf{u} \qquad (4)$$

The coefficients vector **a** can be obtained:

$$\mathbf{a} = \mathbf{A}^{-1}\mathbf{B}\mathbf{u} \qquad (5)$$

where **A** and **B** are defined by:

$$\mathbf{A} = \mathbf{wp}^{\mathrm{T}}\mathbf{p} \qquad (6)$$

$$\mathbf{B} = \mathbf{wp}^{\mathrm{T}} \qquad (7)$$

A is known also as weight moment matrix. Numerical experiments show that the weight moment matrix **A** is invertible for both regular and irregular distribution of knot points. Note that when the weight is chosen as unity, the B-spline collocation method is recovered. In addition, rather than nodes, it is working with knot points or simply knots.

2.1. *High order B-spline basis functions approximation and construction*

B-spline curve is a piecewise polynomial function that is connected continuously by different curve segments, which is contained in the convex hull of its control poly-lines. B-spline has minimal support with respect to a given degree, smoothness and domain partition (de Boor, 2001).

Three important components define the B-spline curves, namely: knot vector, basis or blending functions and control points. Mathematically, the B-spline curve is defined along the parametric interval t as:

$$\mathbf{C}(t) = \sum_{i=1}^{n+1} \mathbf{C}_i \, N_{i,k}(t) \tag{8}$$

where: [$\mathbf{C}_i$: $i = 1, 2, \ldots, n+1$] are the control points, which are in general do not lie on the curve, k is the order of B-spline basis function, meaning that it represents the polynomial of order $p = k - 1$, $N_{i,k}(t)$ is the normalized B-spline basis function, which is described by the order k and a non-decreasing sequence of real numbers in the knot vector Ξ: [t_i: i = 1, . . . , $n+k+1$], satisfying the Schoenberg-Whitney condition and t is the parameterized coordinate.

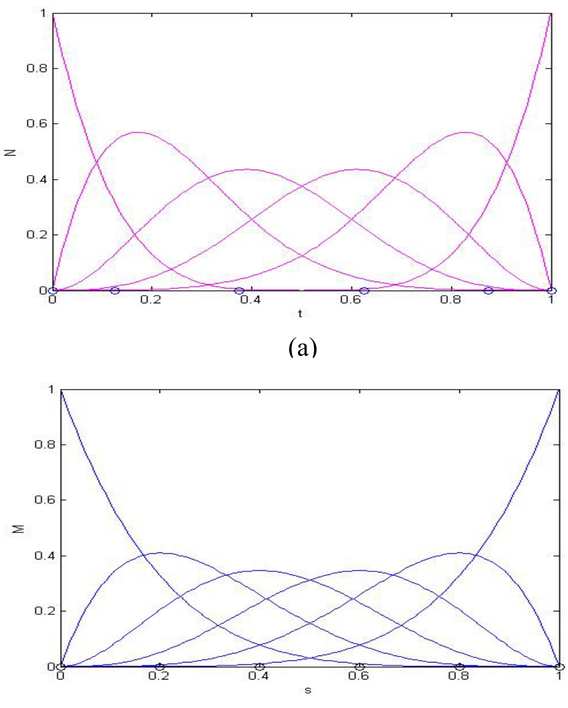


Fig. 1. (a) 5th B-spline basis functions of the knot vector Ξ = [0,0,0,0,0,1/2,1,1,1,1,1] and (b) 6th B-spline basis functions of the knot vector Ξ = [0,0,0,0,0,0,1,1,1,1,1,1].

The Cox-de Boor recursive formula is used to obtain B-spline basis functions of the intended order $N_{i,k}(t)$, starting from the first order B-spline function as follows:

$$N_{i,1}(t) = \begin{cases} 1, t_i \le t < t_{i+1} \\ 0, \text{otherwise} \end{cases} \tag{9}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t) \qquad \text{for } 2 \le k \le n+1 \tag{10}$$

where t_i are the elements of the knot vector previously mentioned. B-spline basis functions have the partition of unity property.

Moreover, the following relationship between the total number of knots in the knot vector m, the control points $n+1$ and the order k holds:

$$m = n + k + 1 \tag{11}$$

Eq. (11) describes that each control point needs basis function which is in the relationship with the knots in the knot vector chosen.

With all the quantities defined, the d-th derivatives of the b-spline can be computed as follows:

$$\mathbf{C}^{(d)}(t) = \sum_{i=1}^{n+1} \mathbf{C}_i \, N^{(d)}{}_{i,k}(t) \tag{12}$$

The application for higher dimensions, 2D or 3D is served by taking the tensor product of the one-dimensional B-splines. For 2D domain, the tensor product is defined as follows:

$$\mathbf{D}(t,s) = \sum_{i=1}^{n+1} \sum_{j=1}^{m+1} \mathbf{D}_{i,j} \, N_{i,k}(t) M_{j,l}(s) \qquad , t_{min} \le t \le t_{max}, s_{min} \le s \le s_{max} \tag{13}$$

where: $N_{i,k}(t)$ and $M_{j,l}(s)$ are the B-spline basis functions of k-th and l-th orders along the parametric directions t and s, respectively, and $\mathbf{D}_{i,j}$ is now the corresponding coefficients of the approximation.

Fig. 1 depicts the B-spline basis functions resulted in from the knot vectors Ξ = [0,0,0,0,0,1/2,1,1,1,1,1] and Ξ = [0,0,0,0,0,0,1,1,1,1,1,1], respectively.

2.2. *Matrix of weight w*

The weight matrix w(**x**) is a distance function defined as:

$$w(\mathbf{x}) = w(\mathbf{x} - \mathbf{x}_j) \tag{14}$$

that plays an important role in the present approach approximation. It smoothes out the approximation produced by giving different influences in the weight moment matrix **A** based upon the distance of data or knot points. The knot points having more distances should give smaller weights, and vice versa.

Arbitrary weight function could be used, but according to (Leitao *et al.*, 2007), it is desirable that the weight w($\mathbf{x}$-$\mathbf{x}_j$) be positive, continuous and smooth and having compact support that it increases monotonically as $\|\mathbf{x}-\mathbf{x}_j\|$ decreases.

In the present paper, the tensor product weights based upon the cubic spline weight function are used for the weight matrix w($\mathbf{x}$-$\mathbf{x}_j$). In two-dimensional application, the tensor product weights are defined as follows:

$$w(\mathbf{x} - \mathbf{x}_j) = w(r_x).w(r_y) \tag{15}$$

in which $w(r)$ is defined as:

$$w(r) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3, \text{for } r \le \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3, \text{for } \frac{1}{2} < r \le 1 \\ 0, \text{for } r \ge 1 \end{cases} \tag{16}$$

r_x and r_y are the Euclidean distance along x- and y-directions, respectively, defined as:

$$r_x = \|x - x_j\|$$
$$r_y = \|y - y_j\| \tag{17}$$

2.3. *Handling Dirichlet and Neumann boundary conditions*

To deal with the application of Dirichlet and Neumann boundary conditions in the boundary value problems considered, the knot vector with *k*-multiple knots, known as the open-uniform knot vector, is chosen in the present study, particularly the one without mid-knot(s). Using the knot vector, B-spline is efficiently designed to have Kronecker delta condition which is useful property for interpolation, thus makes it easy to impose the Dirichlet boundary conditions. The imposition of Neumann boundary conditions is also facilitated easily by the use of the knot vector, such as the open-uniform knot vectors Ξ = [0,0,0,0,0,1/2,1,1,1,1,1] and Ξ = [0,0,0,0,0,0,1,1,1,1,1,1] and their corresponding basis functions in Fig. 1(a) and (b).

3. Collocation Method for Boundary Value Problem

Consider the following general form of a boundary value problem:

$$\begin{aligned} \mathbf{Lu} &= \mathbf{f} \quad in \quad \Omega \\ \mathbf{B}^h\mathbf{u} &= \mathbf{h} \quad on \quad \partial\Omega^h \\ \mathbf{B}^g\mathbf{u} &= \mathbf{g} \quad on \quad \partial\Omega^g \end{aligned} \tag{18}$$

where **Ω** is the problem domain, $\partial\mathbf{\Omega}^{h}$ is the Dirichlet boundary, $\partial\mathbf{\Omega}^{g}$ is the Neumann boundary and piecewise smooth boundary $\partial\mathbf{\Omega} = \partial\mathbf{\Omega}^{h} \cup \partial\mathbf{\Omega}^{g}$.

For Poisson equation, the following notations are introduced: **L** is the Laplace operator, ∇^2, $\mathbf{B}^h = 1$, $\mathbf{B}^g = (\partial / \partial n)$, **n** is the vector of normal directions, whose components n_i, to the Neumann boundary. Furthermore, $\mathbf{f} = f(\mathbf{x},\mathbf{u})$ is the potential function, where $\mathbf{x} \in R^d$ represents the vector of position and *d* is the dimension of the problem domain **Ω**, **h** and **g** are the prescribed values of the Dirichlet and Neumann BCs.

In collocation method, the residuals are enforced to be zeros at a set of collocation points: *Np* is a set of collocation points in Ω, *Nb* is a set of collocation points in $\partial\mathbf{\Omega}^{h}$ and *Nq* is a set of collocation points in $\partial\mathbf{\Omega}^{g}$, thus forming a set of discrete equations defining the boundary value problem.

4. Results and Discussion

Several numerical examples are presented in this section to show the capability of the present approach for solving boundary value problems in both regular and irregular domains. To assess the numerical performance of the approach, this error norm is used:

$$E_n = \sqrt{\frac{\sum_{i=1}^{n}\left[u(x_i) - \bar{u}(x_i)\right]^2}{\sum_{i=1}^{n} u(x_i)^2}} \tag{19}$$

n is the number of collocation points, $u(x_i)$ and $\bar{u}(x_i)$ represent the exact solution and the present method approximation, respectively.

4.1. *Poisson equation in annulus domain*

Consider the following equation:

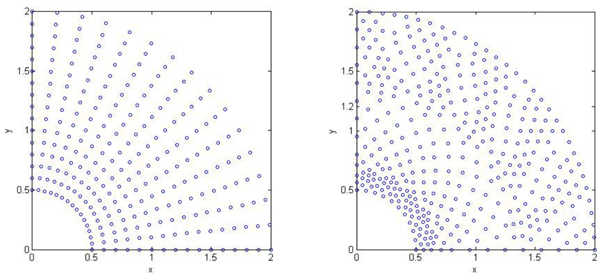
$$\nabla^2 u = -2\pi^2 \sin(\pi x)\sin(\pi y) \tag{20}$$

which is applied on the annulus region with the corresponding Dirichlet boundary conditions given by the exact solution of the Poisson equation as follows:

$$u = \sin(\pi x)\sin(\pi y) \tag{21}$$

Fig. 2 describes the problem domain of annulus geometry with two different arrangements of knot points serving also as collocation points.

The exact solution of the Poisson problem over the annulus domain is depicted in Fig. 3. Table 1 presents the accuracy of the weighted least squares B-spline collocation method when solving for the Poisson problem.



(a) (b)

Fig. 2. Annulus geometry examined with: (a) regular knot points and (b) irregular knot points.

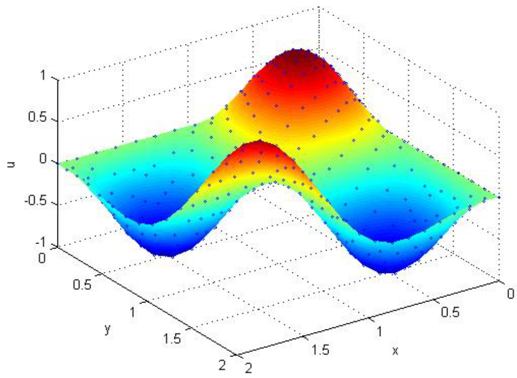


Fig. 3. The exact solution of the Poisson problem over the annulus region (blue points represent the approximation results evaluated at the collocation points as in Fig. 2(a)).

Table 1. Accuracy of the weighted least squares B-spline collocation method for the Poisson problem over the annulus domain

Arrangement of collocation points	Number of collocation points	Error (E_n)
Regular	256	1.5892E-5
Irregular	373	8.6763E-7

4.2. ***Poisson equation in complex domain-I***

The following Poisson equation:

$$\nabla^2 u = \left(\lambda_1{}^2 + \lambda_2{}^2\right)\exp\left(\lambda_1 x + \lambda_2 y\right) \tag{22}$$

is applied on the complex region-I as described in Figs. 4 and 5. The exact solution of the Poisson problem is given by:

$$u = \exp\left(\lambda_1 x + \lambda_2 y\right) \tag{23}$$

where λ_1 and λ_2 are chosen as 0.3 and 0.8, respectively.

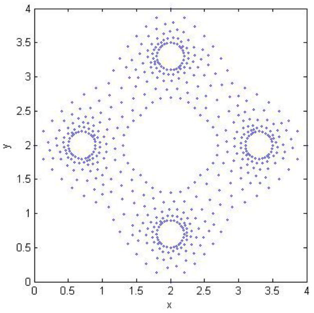


Fig. 4. Complex domain-I along with its knot points distribution.

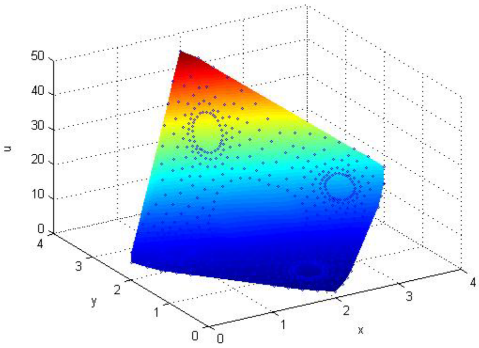


Fig. 5. The exact solution of the Poisson problem over the complex domain-I (blue points represent the approximation results evaluated at the collocation points).

4.3. *Poisson equation in complex domain-II*

Given the Poisson equation:

$$\nabla^2 u = -18\pi^2 \sin(3\pi x)\sin(3\pi y) \tag{24}$$

which is applied on the complex region-II as described in Fig. 6. The exact solution of the Poisson problem is given by:

$$u = \sin(3\pi x)\sin(3\pi y) \tag{25}$$

The accuracy of the weighted least squares B-spline collocation method in solving for the Poisson problem over the complex domain-II is shown in Table 3.

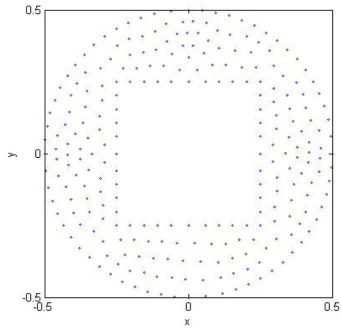


Fig. 6. Complex domain-II with 260 knot points distribution.

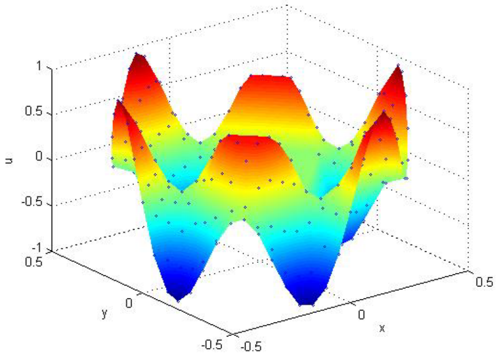


Fig. 7. The exact solution of the Poisson problem over the complex domain-II (blue points represent the approximation results evaluated at the collocation points).

Table 3. Accuracy of the weighted least squares B-spline collocation method for the Poisson problem over the complex domain-II

Problem domain	Number of collocation points	Error (E_n)
Complex domain-II	96	3.1099E-4
	260	2.7774E-5
	526	1.3217E-5
	668	1.2366E-5

4.4. ***Poisson equation with Neumann boundary conditions***

Consider the following Poisson equation:

$$\nabla^2 u = -2\pi^2 \cos(\pi x)\cos(\pi y) \quad (26)$$

defined in Ω = [0, 1] x [0, 1] and subjected to the Dirichlet boundary condition:

$$u(0, y) = \cos(\pi y) \quad \text{on} \quad x = 0 \quad (27)$$

together with the Neumann boundary conditions:

$$\frac{\partial u(1, y)}{\partial x} = 0 \quad \text{on} \quad x = 1 \quad (28a)$$

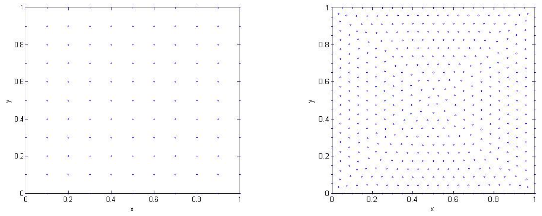
$$\frac{\partial u(x,0)}{\partial y} = 0 \quad \text{on} \quad y = 0 \quad (28b)$$

$$\frac{\partial u(x,1)}{\partial y} = 0 \quad \text{on} \quad y = 1 \quad (28c)$$

The corresponding exact solution is given by:

$$u = \cos(\pi x)\cos(\pi y) \quad (29)$$

Fig. 8 shows the application domain of the Poisson problem with both regular and irregular collocation points. The exact solution of the Poisson problem is further depicted in Fig. 9.



(a) (b)

Fig. 8. Square problem domain of the Poisson equation with Dirichlet and Neumann BCs: (a) regular and (b) irregular collocation points.

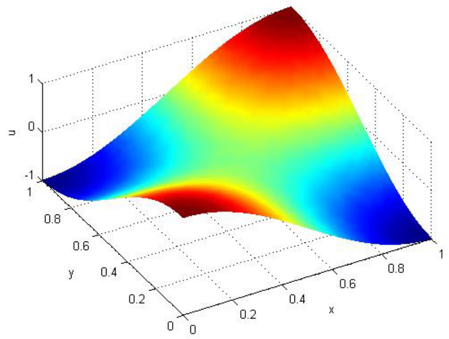


Fig. 9. The exact solution of the Poisson problem with Dirichlet and Neumann BCs over the square domain.

The accuracies obtained by the weighted least squares B-spline collocation method for the solution of the Poisson problem are shown in Table 4.

Table 4. Accuracy of the weighted least squares B-spline collocation method for the Poisson problem with Dirichlet and Neumann BCs

Arrangement of collocation points	Number of collocation points	Error (E_n)
Regular	6x6	9.4114E-3
	11x11	2.7594E-6
	22x22	1.9389E-8
	24x24	1.7318E-11
Irregular	488	4.2148E-7

4.5. *Approximation order and convergence studies*

There is great flexibility when working with B-spline in particular with respect to its approximation order as shown by the Cox-de Boor recursive formula i.e. Eqs. (9) and (10). From the numerical examples considered in the present study, it was found that the B-spline approximation of ≥ 10th is suitable to use with. It is important to note, however, that the use of B-spline with much higher order could introduce the further computational cost. To further describe the accuracies obtained by the present approach, the corresponding convergence analysis is also presented. Figs. 10 and 11 depict the convergence rates of the present approach for the Poisson problems in complex domain-II and square domain, respectively. It is clear that the present approach has high convergence rates for the Poisson problems with Dirichlet and Neumann boundary conditions along with regular and irregular points/nodes.

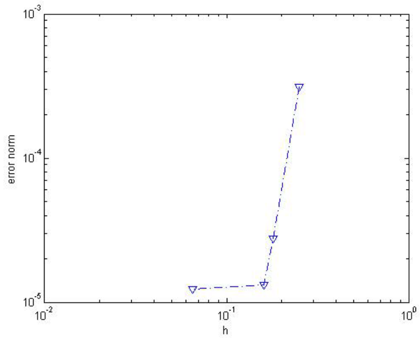


Fig. 10. The convergence rate for the Poisson problem in complex domain-II.

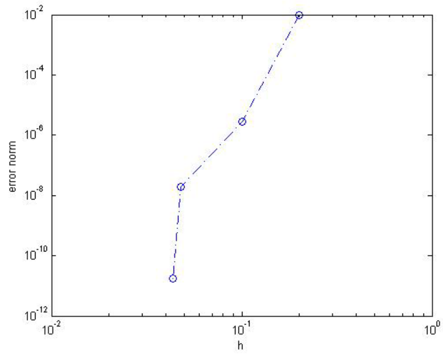


Fig. 11. The convergence rate for the Poisson problem with Dirichlet and Neuman boundary conditions in square domain of regular nodes.

5. Conclusions

In this paper, a weighted least squares B-spline collocation method has been introduced for solving Poisson problems. It requires free-integration over the problem domain and introduces no difficulties in the imposition of both Dirichlet and Neumann boundary conditions. It also possesses high convergence rates for both the Poisson problems. It is hence applicable and suitable for arbitrary domain applications. Further research is needed to improve and extend its applications in various problems.

References

Atluri, S.N. and Zhu, T. (1998), "A new meshless local Petrov–Galerkin (MLPG) approach in computational mechanics", Comput. Mech. Vol. 22, pp. 117–127.

Bathe, K.J. (1996), "Finite Element Procedures", Prentice Hall, Inc., New Jersey, USA.

Belytschko, T., Lu, Y.Y. and Gu, L. (1994), "Element-free Galerkin methods", Int. J. Numer. Meth. Eng. Vol. 37, pp. 229–256.

Boroomand, B., Najjar, M. and Onate, E. (2009), "The generalized finite point method", Comput. Mech. Vol. 44, pp. 173–190.

Botella, O. (2002), "On a collocation B-spline method for the solution of the Navier–Stokes equations", Comp. Fluids. Vol. 31, pp. 397-420.
Burla, R.K. and Kumar, A.V. (2008), "Implicit boundary method for analysis using uniform B-spline basis and structured grid", Int. J. Numer. Meth. Eng. Vol. 76, pp. 1993-2028.
Chaniotis, A.K. and Poulikakos, D. (2004), "High order interpolation and differentiation using B-splines," Journal of Computational Physics Vol. 197, pp. 253–274.
Chui, C.K. (1988), "Multivariate Splines", second ed., SIAM, Philadelphia, USA.
de Boor, C. (2001), "A Practical Guide to Splines", revised ed., Springer, New York, USA.
Dehghan, M. and Ghesmati, A. (2010), "Combination of meshless local weak and strong (MLWS) forms to solve the two dimensional hyperbolic telegraph equation", Eng. Anal. Bound. Elem. Vol. 34, pp. 324–336.
Farin, G. (2002), "Curves and Surfaces for Computer Aided Geometric Design", fifth ed., Academic Press, San Diego, CA.
Haykin, S. (2009), "Neural Networks and Learning Machines", 3rd ed., Pearson Prentice Hall, USA.
Hoggar, S.G. (2006), "Mathematics of Digital Images", first ed., Cambridge University Press, Cambridge.
Hollig, K. (2003), "Finite Element Methods with B-splines", SIAM, Philadelphia, USA.
Jator, S. and Sinkala, Z. (2007), "A high order B-spline collocation method for linear boundary value problems", App. Math. Comp. Vol. 191, pp. 100–116.
Johnson, R.W. (2005), "Higher order B-spline collocation at the Greville abscissae", App. Num. Math. Vol. 52, pp. 63–75.
Leitao, V.M.A., Alves, C.J.S. and Duarte, C.A. (Eds.) (2007), "Advances in Meshfree Techniques", Springer, Netherlands.
Liu, G.R. and Gu, Y.T. (2003), "A meshfree method: meshfree weak–strong (MWS) form method, for 2-D solids", Comp. Mech. Vol. 33, pp. 2-14.
Liu, X. and Tai, K. (2006), "Point interpolation collocation method for the solution of partial differential equations", Eng. Anal. Bound. Elem. Vol. 30, pp. 598–609.
Liu, X., Liu, G.R., Tai, K. and Lam, K.Y. (2005), "Radial point interpolation collocation method (RPICM) for partial differential equations", Comp. Math. App. Vol. 50, pp. 1425-1442.
Nocedal, J. and Wright, S.J. (2006), "Numerical Optimization", 2nd ed., Springer, New York.
Ohayon, R. (2004), Fluid–Structure Interaction Problem, in: E. Stein, R. de Borst, T.J.R. Hugues (Eds.), Encyclopedia of Computational Mechanics, Vol. 2, John Wiley, pp. 683–694.
Salomon, D. (2006), "Curves and Surfaces for Computer Graphics", Springer Science + Business Media, Inc. New York, USA.
Wang, H. and Qin, Q.H. (2006), "A meshless method for generalized linear or nonlinear Poisson-type problems", Eng. Anal. Bound. Elem. Vol. 30, pp. 515–521.
Wolberg, J. (2006), "Data Analysis Using the Method of Least Squares", Springer-Verlag, Berlin.
Yu, K.H., Kadarman, A.H. and Djojodihardjo, H. (2010), "Development and implementation of some BEM variants-A critical review", Eng. Anal. Bound. Elem. Vol. 34, pp. 884–899.
Zhang, X., Song, K.Z., Lu, M.W. and Liu, X. (2000), "Meshless methods based on collocation with radial basis functions", Comput. Mech. Vol. 26, pp. 333–343.